INSTITUT D'ASTROPHYSIQUE DE PARIS
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Multimessengers @ Prague

Multimessengers, Compact Objects and Fundamental Physics

HIGH-ORDER POST-NEWTONIAN CALCULATIONS FOR GRAVITATIONAL WAVES

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Gravitation et Cosmologie ($\mathcal{G}\mathbb{R}\varepsilon\mathbb{C}\mathcal{O}$)
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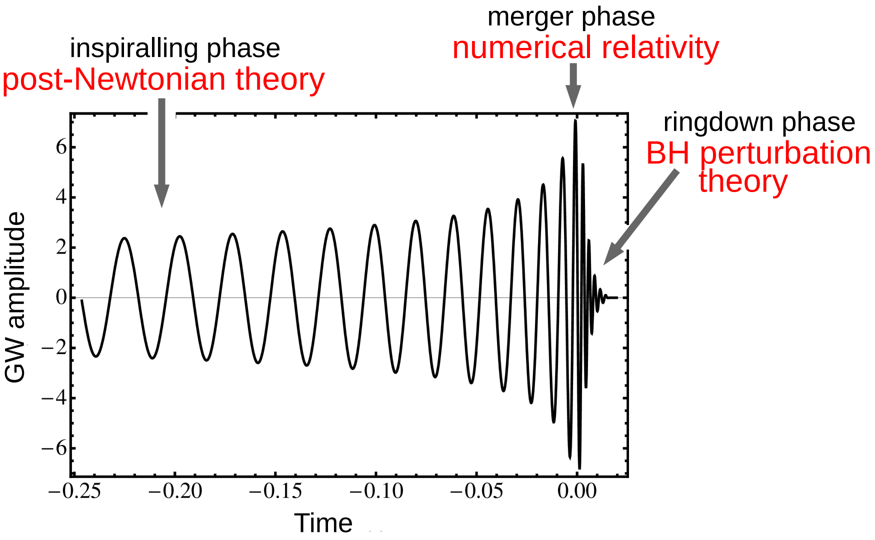
Plan of the talk

1. Methods to compute GW templates
2. Gravitational wave generation formalism
3. Problem of the 4PN equations of motion
4. Progress on the 4.5PN GW generation
5. Radiation reaction and flux-balance equations

METHODS TO COMPUTE GW TEMPLATES

The gravitational chirp of binary black holes



Quadrupole moment formalism [Einstein 1918; Landau & Lifchitz 1945]

$$4\pi R^2 \bar{\mathfrak{J}} = \frac{\varkappa}{40\pi}\left[\sum_{\mu\nu} \dddot{J}_{\mu\nu}^{2} - \frac{1}{3}\left(\sum_{\mu} \dddot{J}_{\mu\mu}\right)^2\right].$$

1. Einstein quadrupole formula

$$\boxed{\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)^{\text{GW}} = \frac{G}{5c^5}\left\{\frac{\mathrm{d}^3 I_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 I_{ij}}{\mathrm{d}t^3} + \mathcal{O}\left(\frac{v}{c}\right)^2\right\}}$$

2. Amplitude quadrupole formula

$$\boxed{h_{ij}^{\text{TT}} = \frac{2G}{c^4 R}\left\{\frac{\mathrm{d}^2 I_{ij}}{\mathrm{d}t^2}\left(t - \frac{R}{c}\right) + \mathcal{O}\left(\frac{v}{c}\right)\right\}^{\text{TT}} + \mathcal{O}\left(\frac{1}{R^2}\right)}$$

3. Radiation reaction formula [Chandrasekhar & Esposito 1970; Burke & Thorne 1970]

$$\boxed{F_i^{\text{reac}} = -\frac{2G}{5c^5}\rho\, x^j \frac{\mathrm{d}^5 I_{ij}}{\mathrm{d}t^5} + \mathcal{O}\left(\frac{v}{c}\right)^7}$$

which is a 2.5PN $\sim (v/c)^5$ effect in the source's equations of motion

Radiation reaction and balance equations

1. Conserved Newtonian energy in the source

$$E = \int \mathrm{d}^3\mathbf{x}\,\rho\left[\frac{\mathbf{v}^2}{2} + \Pi - \frac{U}{2}\right]$$

2. Eulerian equations of motion in the source

$$\rho\frac{\mathrm{d}v^i}{\mathrm{d}t} = -\partial_i P + \rho\partial_i U - \overbrace{\frac{2G}{5c^5}\rho\, x^j\,\frac{\mathrm{d}^5 I_{ij}}{\mathrm{d}t^5}}^{\boldsymbol{F}^{\text{reac}}}$$

3. Energy loss is due to the work of the radiation reaction force

$$\boxed{\frac{\mathrm{d}E}{\mathrm{d}t} = \int \mathrm{d}^3\mathbf{x}\,\boldsymbol{v}\cdot\boldsymbol{F}^{\text{reac}} = -\frac{G}{5c^5}\frac{\mathrm{d}^3 I_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 I_{ij}}{\mathrm{d}t^3} + \text{total time derivative}}$$

4. Obtain the balance equation after averaging over one period

$$\langle\frac{\mathrm{d}E}{\mathrm{d}t}\rangle = -\langle\mathcal{F}^{\text{GW}}\rangle$$

Summary of known PN orders

Method	Equations of motion	Energy flux	Waveform
Multipolar-post-Minkowskian & post-Newtonian (MPM-PN)	4PN 3.5PN (NNL) SO 3PN (NL) SS 3.5PN (NL) SSS	3.5PN + 4.5PN 4PN (NNL) SO 3PN (NL) SS 3.5PN (NL) SSS	3.5PN 1.5PN (L) SO 2PN (L) SS
Canonical ADM Hamiltonian	4PN 3.5PN (NNL) SO 4PN (NNL) SS 3.5PN (NL) SSS	1PN	
Effective Field Theory (EFT)	4PN 2.5PN (NL) SO 4PN (NNL) SS	2PN 3PN (NL) SS	
Direct Integration of Relaxed Equations (DIRE)	2.5PN 1.5PN (L) SO 2PN (L) SS	2PN 1.5PN (L) SO 2PN (L) SS	2PN 1.5PN (L) SO 2PN (L) SS
Surface Integral	3PN		

Many works devoted to spins:

- Spin effects (SO, SS, SSS) are known in EOM up to 4PN order
- SO effects are known in radiation field up to 4PN
- SS in radiation field known to 3PN

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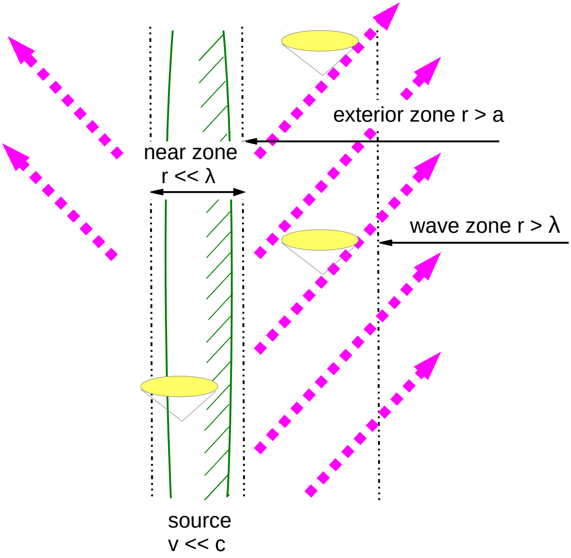
GRAVITATIONAL WAVE GENERATION FORMALISM

Regions of space around the GW source



Linearized multipolar vacuum solution [Pirani 1964; Thorne 1980]

Solution of linearized vacuum field equations in harmonic coordinates

$$\boxed{\Box h^{\alpha\beta}_{(1)} = \partial_\mu h^{\alpha\mu}_{(1)} = 0}$$

$$h^{00}_{(1)} = -\frac{4}{c^2}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_L\left(\frac{1}{r}I_L\right) \qquad \boxed{L = i_1 i_2 \cdots i_\ell}$$

$$h^{0i}_{(1)} = \frac{4}{c^3}\sum_{\ell=1}^{+\infty}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-1}\left(\frac{1}{r}I^{(1)}_{iL-1}\right) + \frac{\ell}{\ell+1}\varepsilon_{iab}\partial_{aL-1}\left(\frac{1}{r}J_{bL-1}\right)\right\}$$

$$h^{ij}_{(1)} = -\frac{4}{c^4}\sum_{\ell=2}^{+\infty}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-2}\left(\frac{1}{r}I^{(2)}_{ijL-2}\right) + \frac{2\ell}{\ell+1}\partial_{aL-2}\left(\frac{1}{r}\varepsilon_{ab(i}J^{(1)}_{j)bL-2}\right)\right\}$$

- multipole moments $I_L(u)$ and $J_L(u)$ are arbitrary functions of $u = t - r/c$
- mass $M \equiv I = \mathrm{const}$, center-of-mass position $G_i \equiv I_i = \mathrm{const}$
  linear momentum $P_i \equiv I^{(1)}_i = 0$, angular momentum $J_i = \mathrm{const}$

Multipolar-post-Minkowskian expansion

[Blanchet & Damour 1986, 1988, 1992; Blanchet 1987, 1993, 1998]

1. Look for the general multipolar expansion $\mathcal{M}(h)$ generated outside the source in the form [Bonnor 1959, Bonnor & Rotenberg 1961]

$$\mathcal{M}(h) = \underbrace{G\,h_{(1)} + G^2\,h_{(1)} + \cdots + G^n\,h_{(n)} + \cdots}_{\text{formal post-Minkowskian expansion}}$$

2. Start from the previous general multipolar solution $h^{\alpha\beta}_{(1)}$ of the vacuum field equation at the linear order

3. Iterate that solution using a regularization scheme based on A.C. in $B \in \mathbb{C}$ to cope with the singularity of the multipole expansion when $r \to 0$

$$\mathop{\text{Finite Part}}_{B=0} \Box^{-1}_{\text{ret}} \left[(r/r_0)^B f \right]$$

Multipolar-post-Minkowskian expansion

[Blanchet & Damour 1986, 1988, 1992; Blanchet 1987, 1993, 1998]

1. At n-th post-Minkowskian order we need to solve

$$\partial_\nu h^{\mu\nu}_{(n)} = 0$$
$$\Box h^{\mu\nu}_{(n)} = \Lambda^{\mu\nu}\big(\underbrace{h_{(1)}, \cdots h_{(n-1)}}_{\text{known from previous iterations}}\big)$$

2. A particular solution with the required multipole structure reads

$$u^{\mu\nu}_{(n)} = \mathop{\mathrm{FP}}_{B=0} \Box^{-1}_{\text{Ret}} \left[\left(\frac{r}{r_0}\right)^B \Lambda^{\mu\nu}_{(n)} \right]$$

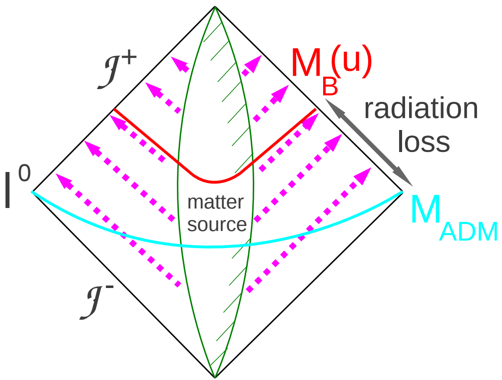
3. An homogeneous solution is added to guarantee the harmonic gauge condition

$$\boxed{h^{\mu\nu}_{(n)} = u^{\mu\nu}_{(n)} + v^{\mu\nu}_{(n)}}$$

4. The MPM solution is represents the most general solution of Einstein's vacuum equations outside an isolated matter system

Asymptotic structure of radiating space-time

[Bondi-Sachs-Penrose formalism 1960s]



$$M_{\mathsf{B}}(u) = M_{\mathsf{ADM}} - \overbrace{\frac{G}{5c^7}\int_{-\infty}^{u} \mathrm{d}t\, U^{(1)}_{ij}(t) U^{(1)}_{ij}(t)}^{\text{mass-energy emitted in GW}} + \begin{cases} \text{higher multipole moments} \\ \text{higher PN approximations} \end{cases}$$

Problem of the matching

[Lagerström *et al.* 1967; Burke & Thorne 1971; Kates 1980; Anderson *et al.* 1982; Blanchet 1998]

1. Most general multipolar(-post-Minkowskian) solution in the source's exterior

$$\mathcal{M}(h) = \underset{B=0}{\mathrm{FP}}\,\Box^{-1}_{\text{ret}}\left[\left(\frac{r}{r_0}\right)^B \mathcal{M}(\Lambda)\right] + \sum_{\ell=0}^{+\infty} \partial_L \left\{ \frac{I_L(t-r/c)}{r} \right\}$$

where the homogeneous solution is parametrized by multipole moments

2. Most general PN solution in the source's near zone

$$\bar{h} = \underset{B=0}{\mathrm{FP}}\,\Box^{-1}_{\text{sym}}\left[\left(\frac{r}{r_0}\right)^B \bar{\tau}\right] + \sum_{\ell=0}^{+\infty} \partial_L \left\{ \frac{R_L(t-r/c) - R_L(t+r/c)}{r} \right\}$$

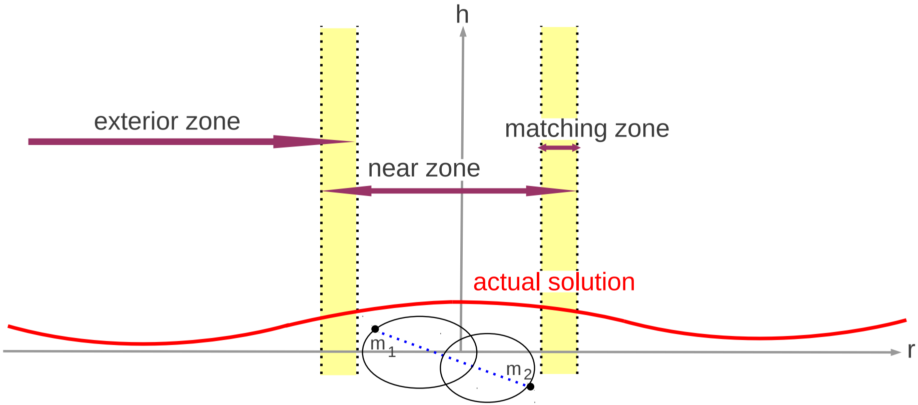
where the homogeneous solution (regular when $r \to 0$) is parametrized by "radiation reaction" multipole moments

3. We impose the matching equation

$$\boxed{\overline{\mathcal{M}(h)} = \mathcal{M}(\bar{h})}$$

Problem of the matching

[Lagerström *et al.* 1967; Burke & Thorne 1971; Kates 1980; Anderson *et al.* 1982; Blanchet 1998]



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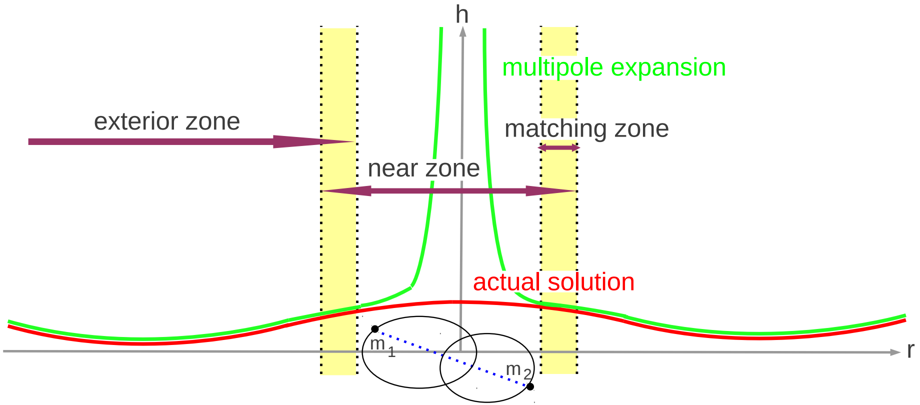

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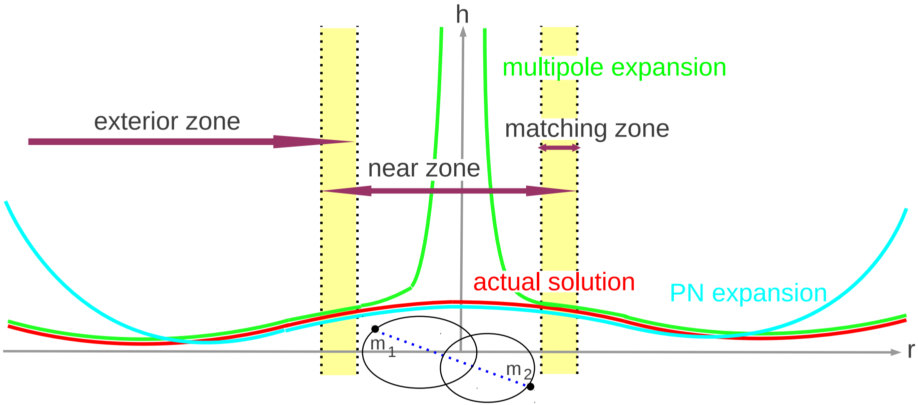

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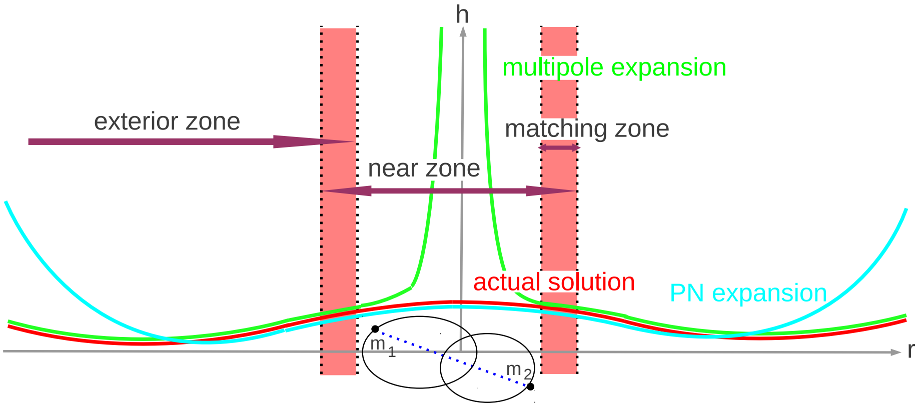


General solution of the matching equation

1. In the far zone

$$\boxed{\mathcal{M}(h) = \mathop{\mathrm{FP}}_{B=0} \Box^{-1}_{\text{ret}} \left[\left(\frac{r}{r_0}\right)^B \mathcal{M}(\Lambda) \right] - \frac{4G}{c^4} \underbrace{\sum_{\ell=0}^{+\infty} \partial_L \left\{ \frac{\mathcal{F}_L(t-r/c)}{r} \right\}}_{\text{source's multipole moments}}}$$

2. In the near zone [Poujade & Blanchet 2002; Blanchet, Faye & Nissanke 2005]

$$\boxed{\bar{h} = \mathop{\mathrm{FP}}_{B=0} \Box^{-1}_{\text{ret}} \left[\left(\frac{r}{r_0}\right)^B \bar{\tau} \right] - \frac{4G}{c^4} \underbrace{\overline{\sum_{\ell=0}^{+\infty} \partial_L \left\{ \frac{\mathcal{R}_L(t-r/c) - \mathcal{R}_L(t+r/c)}{r} \right\}}}_{\text{non-local tail term (4PN order)}}}$$

PROBLEM OF THE 4PN EQUATIONS OF MOTION

4PN equations of motion of compact binary systems

$$\frac{\mathrm{d}\boldsymbol{v}_1}{\mathrm{d}t} = -\frac{G m_2}{r_{12}^2}\boldsymbol{n}_{12} + \overbrace{\frac{1}{c^2}\left\{\left[\frac{5G^2 m_1 m_2}{r_{12}^3} + \frac{4G^2 m_2^2}{r_{12}^3} + \cdots\right]\boldsymbol{n}_{12} + \cdots\right\}}^{\text{1PN Lorentz-Droste-Einstein-Infeld-Hoffmann term}} + \underbrace{\frac{1}{c^4}\left[\cdots\right]}_{\text{2PN}} + \underbrace{\frac{1}{c^5}\left[\cdots\right]}_{\substack{\text{2.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^6}\left[\cdots\right]}_{\text{3PN}} + \underbrace{\frac{1}{c^7}\left[\cdots\right]}_{\substack{\text{3.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^8}\left[\cdots\right]}_{\substack{\text{4PN}\\ \text{conservative \& radiation tail}}} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

3PN	[Jaranowski & Schäfer 1999; Damour, Jaranowski & Schäfer 2001ab]	ADM Hamiltonian
	[Blanchet-Faye-de Andrade 2000, 2001; Blanchet & Iyer 2002]	Harmonic EOM
	[Itoh & Futamase 2003; Itoh 2004]	Surface integral method
	[Foffa & Sturani 2011]	Effective field theory
4PN	[Jaranowski & Schäfer 2013; Damour, Jaranowski & Schäfer 2014]	ADM Hamiltonian
	[Bernard, Blanchet, Bohé, Faye, Marchand & Marsat 2015, 2016, 2017abc]	Fokker Lagrangian
	[Foffa & Sturani 2013, 2019; Foffa, Porto, Rothstein & Sturani 2019]	Effective field theory

The Fokker Lagrangian approach to the 4PN EOM

Based on collaborations with





Laura Bernard, **Alejandro Bohé**, **Guillaume Faye**,
Tanguy Marchand & **Sylvain Marsat**

[PRD **93**, 084037 (2016); **95**, 044026 (2017); **96**, 104043 (2017); **97**, 044023 (2018); **97**, 044037 (2018)]

Fokker action of N point particles

- Gauge-fixed Einstein-Hilbert action of N point particles

$$S_{\mathsf{EH}} = \frac{c^3}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\Big[R \underbrace{-\frac{1}{2}\Gamma^\mu\Gamma_\mu}_{\text{gauge-fixing term}}\Big] - \sum_a \underbrace{m_a c^2 \int \mathrm{d}\tau_a}_{N \text{ point particles}}$$

- The Fokker PN action is obtained by inserting an explicit iterated PN solution of the Einstein field equations

$$g_{\mu\nu}(\mathbf{x},t) \longrightarrow \overline{g}_{\mu\nu}(\mathbf{x};\boldsymbol{x}_a(t),\boldsymbol{v}_a(t),\cdots)$$

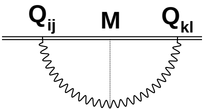
- The PN equations of motion of the N particles (self-gravitating system) are

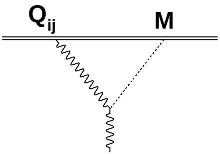
$$\frac{\delta S_{\mathsf{F}}}{\delta \boldsymbol{x}_a} \equiv \frac{\partial L_{\mathsf{F}}}{\partial \boldsymbol{x}_a} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{\mathsf{F}}}{\partial \boldsymbol{v}_a}\right) + \cdots = 0$$

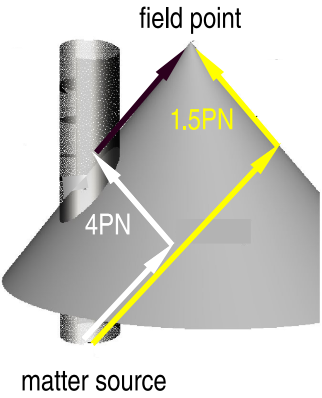
- The Fokker action is equivalent to the effective action of the EFT

The gravitational wave tail effect

[Blanchet & Damour 1988; Blanchet 1993, 1997; Foffa & Sturani 2011; Galley, Leibovich, Porto & Ross 2016]







- In the near zone (4PN effect)

$$\boxed{S^{\text{tail}} = \frac{G^2 M}{5c^8}\,\text{Pf}\iint \frac{\mathrm{d}t\mathrm{d}t'}{|t-t'|}\,Q^{(3)}_{ij}(t)\,Q^{(3)}_{ij}(t')}$$

- In the far zone (1.5PN effect)

$$\boxed{h^{\text{tail}}_{ij} = \frac{4G}{c^4 r}\frac{GM}{c^3}\int_{-\infty}^{t}\mathrm{d}t'\,Q^{(4)}_{ij}(t')\ln\left(\frac{t-t'}{\tau_0}\right)}$$

Potential modes versus radiation modes

- The potential modes are responsible for conservative near zone effect and can be computed with the symmetric propagator (when neglecting radiation reaction effects)
- The radiation modes are conservative effects coming from gravitational waves propagating at infinity and re-expanded in the near zone. The first radiation effect in the Fokker action is the non local tail effect at 4PN order
- To high PN order there is a complicated mix up between potential and radiation modes encapsuled in the general formula

$$\bar{h} = \underbrace{\mathop{\mathrm{FP}}_{B=0} \Box^{-1}_{\text{ret}} \left[\left(\frac{r}{r_0}\right)^B \bar{\tau} \right]}_{\text{potential modes}} - \underbrace{\frac{4G}{c^4} \sum_{\ell=0}^{+\infty} \partial_L \left\{ \frac{\mathcal{R}_L(t - r/c) - \mathcal{R}_L(t + r/c)}{r} \right\}}_{\text{radiation modes}}$$

Dimensional regularization of the Fokker action

- UV divergences due to the modelling of compact objects by point particles plague the potential modes starting from the 3PN order
- IR divergences in the Einstein-Hilbert part of the Fokker action (potential modes) occur at the 4PN order
- The IR pole in the potential modes should be compensated by an UV pole coming from the non-local tail term at 4PN order (radiation mode)
- UV and IR divergences are treated with dimensional regularization ($d = 3 + \varepsilon$)

$$\boxed{\begin{aligned} G_{\text{ret}}(\mathbf{x},t) &= -\frac{\tilde{k}}{4\pi}\frac{\theta(t-r)}{r^{d-1}}\,\gamma_{\frac{1-d}{2}}\left(\frac{t}{r}\right) \\ \gamma_s(z) &= \frac{2\sqrt{\pi}}{\Gamma(s+1)\Gamma(-s-\frac{1}{2})}\left(z^2-1\right)^s \quad \text{(such that } \int_1^{+\infty}\mathrm{d}z\,\gamma_s(z) = 1\text{)} \end{aligned}}$$

- The regularization is followed with a renormalization by particle's shifts

Potential mode contribution to IR divergences

- The Hadamard regularization of IR divergences reads

$$I_{\mathcal{R}}^{\mathsf{HR}} = \mathop{\mathrm{FP}}_{B=0} \int_{r>\mathcal{R}} \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B F(\mathbf{x})$$

- The corresponding dimensional regularization reads

$$I_{\mathcal{R}}^{\mathsf{DR}} = \int_{r>\mathcal{R}} \frac{\mathrm{d}^d\mathbf{x}}{\ell_0^{d-3}} F^{(d)}(\mathbf{x})$$

- The difference between the two regularization is of the type ($\varepsilon = d-3$)

$$\boxed{\mathcal{D}I = \sum_q \Bigg[\underbrace{\frac{1}{(q-1)\varepsilon}}_{\text{IR pole}} - \ln\left(\frac{r_0}{\ell_0}\right)\Bigg] \int \mathrm{d}\Omega_{2+\varepsilon}\, \varphi_{3,q}^{(\varepsilon)}(\mathbf{n}) + \mathcal{O}(\varepsilon)}$$

UV divergences coming from the radiation mode

1. At 4PN order the radiation mode is due to the presence of the tail effect described in 3 dimensions by

$$\mathcal{H}_{\text{tail}} = -\frac{4G}{c^4}\overline{\sum_{\ell=0}^{+\infty}\partial_L\left\{\frac{\mathcal{R}_L(t-r/c)-\mathcal{R}_L(t+r/c)}{r}\right\}}$$

2. In d dimensions it reads

$$\mathcal{H}_{\text{tail}} = \sum_{\ell=0}^{+\infty}\sum_{k=0}^{+\infty}\frac{1}{c^{2k}}\,\Delta^{-k}\hat{x}_L\,f_L^{(2k)}(t)$$

$$f_L(t) = \frac{(-)^{\ell+1}\tilde{k}}{4\pi\ell!}\,\underset{B=0}{\mathrm{FP}}\int_1^{+\infty}\mathrm{d}z\,\gamma_{\frac{1-d}{2}}(z)\int\mathrm{d}^d\mathbf{x}\left(\frac{r}{r_0}\right)^B\hat{\partial}_L\left[\frac{\mathcal{M}(\Lambda)(\mathbf{y},t-zr/c)}{r^{d-2}}\right]_{\mathbf{y}=\mathbf{x}}$$

3. In intermediate calculations of radiation modes it is important to keep the parameter B and apply first the limit $B\to 0$ for any $\varepsilon>0$

The B-ε regularization scheme

1. Specializing to the quadratic mass quadrupole interaction $F_L \sim M \times Q_{ij}$ the multipolar source term will itself be of the type

$$\mathcal{M}(\Lambda)_L(r, t - zr/c) = r^{-2d+6-k} \int_1^{+\infty} \mathrm{d}y\, y^p\, \gamma_{\frac{1-d}{2}}(y) F_L(t - (y+z)r/c)$$

2. After a series of transformations we end up with

$$f_L = \underset{B=0}{\mathrm{FP}} \frac{(-)^{\ell+k}\, C_\ell^{p,k}(\varepsilon, B)}{2\ell+1+\varepsilon} \frac{\Gamma(2\varepsilon - B)}{\Gamma(\ell+k-1+2\varepsilon-B)} \int_0^{+\infty} \mathrm{d}\tau\, \tau^{B-2\varepsilon}\, F_L^{(\ell+k-1)}(t-\tau)$$

3. The numerical coefficient is defined by analytic continuation in B and ε

$$C_\ell^{p,k}(\varepsilon, B) = \int_1^{+\infty} \mathrm{d}y\, y^p\, \gamma_{-1-\frac{\varepsilon}{2}}(y) \int_1^{+\infty} \mathrm{d}z\, (y+z)^{\ell+k-2+2\varepsilon-B} \gamma_{-\ell-1-\frac{\varepsilon}{2}}(z)$$

4. The regulator B is needed to protect against the divergence of this integral at infinity (when $y, z \to +\infty$, with $y \sim z$)

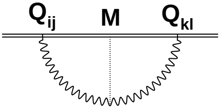
Ambiguity-free completion of the 4PN EOM

1. From the metric we obtain the equations of motion and then identify the corresponding gauge invariant term in the Fokker action
2. We find that the limit $B \to 0$ is finite (no poles) and we obtain a simple closed-form expression for the tail term in an arbitrary d dimension

$$S_{\mathsf{F}}^{\text{tail}} = K_d \, \frac{G^2 M}{c^8} \iint \frac{\mathrm{d}t\,\mathrm{d}t'}{|t-t'|^{2d-5}} Q_{ij}^{(3)}(t)\, Q_{ij}^{(3)}(t')$$

$$\text{with} \quad K_d = \frac{12-12d+5d^2-4d^3+d^4}{8(d-1)^2(d+2)} \left(\frac{2\ell_0^2}{\pi}\right)^{d-3} \frac{\Gamma(-\frac{d}{2})}{\Gamma(\frac{7}{2}-d)\Gamma(\frac{5}{2}-\frac{d}{2})}$$

3. This should correspond exactly to the (real-space version of the) Feynman diagram computed in Fourier space by the EFT community [Galley, Leibovich, Porto & Ross 2016]



Ambiguity-free completion of the 4PN EOM

1. In the limit $\varepsilon \to 0$ this gives Hadamard's "Partie finie" (Pf) integral

$$\boxed{\begin{aligned} S_{\mathsf{F}}^{\text{tail}} &= \frac{G^2 M}{5c^8}\,\underset{\tau_0}{\mathrm{Pf}} \iint \frac{\mathrm{d}t\,\mathrm{d}t'}{|t-t'|} Q_{ij}^{(3)}(t)\,Q_{ij}^{(3)}(t') \\ \text{with}\quad \tau_0 &= \frac{\ell_0}{c\sqrt{\pi}}\exp\Bigg[\underbrace{\frac{1}{2\varepsilon}}_{\text{UV type pole}} - \frac{1}{2}\gamma_{\mathsf{E}} - \frac{41}{60}\Bigg] \end{aligned}}$$

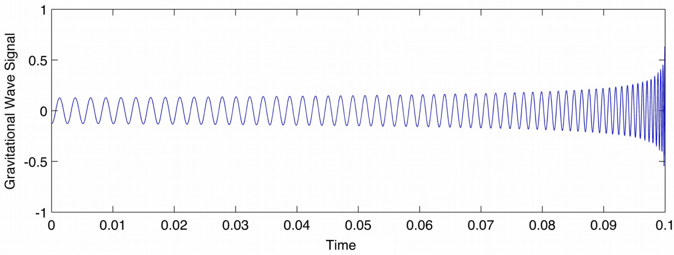
2. We find that the UV pole exactly cancels the IR pole coming from the potential (Einstein-Hilbert) part of the Fokker action
3. Adding up all contributions we obtain the complete EOM at 4PN order with self-consistent derivation of previously conjectured "ambiguity" parameters
4. Recently the EFT approach has also succeeded in a full self-consistent ambiguity-free derivation of the 4PN EOM [Foffa & Sturani 2019; Foffa, Porto, Rothstein & Sturani 2019]
5. The three methods (ADM Hamiltonian, Fokker Lagrangian, EFT) are in perfect agreement on the final result

PROGRESS ON THE 4.5PN GW GENERATION

PN parameters in the orbital phase evolution

$$\phi = \int \omega \, \mathrm{d}t \qquad x = \left(\frac{GM\omega}{c^3}\right)^{2/3} \qquad \nu = \frac{m_1 m_2}{(m_1+m_2)^2}$$

$$\boxed{\phi(x) = \phi_0 - \frac{x^{-5/2}}{32\nu} \sum_p \left(\varphi_{p\text{PN}}(\nu) + \varphi^{(l)}_{p\text{PN}}(\nu)\,\ln x\right) x^p}$$



3.5PN parameters

$$
\begin{aligned}
\varphi_{0\mathrm{PN}} &= 1 \\
\varphi_{1\mathrm{PN}} &= \frac{3715}{1008} + \frac{55}{12}\nu \\
\varphi_{1.5\mathrm{PN}} &= -10\pi \\
\varphi_{2\mathrm{PN}} &= \frac{15293365}{1016064} + \frac{27145}{1008}\nu + \frac{3085}{144}\nu^2 \\
\varphi^{(l)}_{2.5\mathrm{PN}} &= \left(\frac{38645}{1344} - \frac{65}{16}\nu\right)\pi \\
\varphi_{3\mathrm{PN}} &= \frac{12348611926451}{18776862720} - \frac{160}{3}\pi^2 - \frac{1712}{21}\gamma_\mathrm{E} - \frac{3424}{21}\ln 2 \\
&\quad + \left(-\frac{15737765635}{12192768} + \frac{2255}{48}\pi^2\right)\nu + \frac{76055}{6912}\nu^2 - \frac{127825}{5184}\nu^3 \\
\varphi^{(l)}_{3\mathrm{PN}} &= -\frac{856}{21} \\
\varphi_{3.5\mathrm{PN}} &= \left(\frac{77096675}{2032128} + \frac{378515}{12096}\nu - \frac{74045}{6048}\nu^2\right)\pi
\end{aligned}
$$

Toward 4.5PN parameters

1. The 4PN term is only known from perturbative BH theory in the limit $\nu \to 0$ [Tagoshi & Sasaki 1994; Tanaka, Tagoshi & Sasaki 1996]

$$\boxed{\begin{aligned}\varphi_{\mathrm{4PN}} &= \frac{2550713843998885153}{2214468081745920} - \frac{45245}{756}\pi^2 - \frac{9203}{126}\gamma_{\mathrm{E}} - \frac{252755}{2646}\ln 2 \\ &\quad - \frac{78975}{1568}\ln 3 + \mathcal{O}(\nu) \\ \varphi^{(l)}_{\mathrm{4PN}} &= -\frac{9203}{252} + \mathcal{O}(\nu)\end{aligned}}$$

2. The 4.5PN term is known and due to the 4.5PN tail-of-tail-of-tail integral for circular orbits [Marchand, Blanchet & Faye 2017]

$$\boxed{\begin{aligned}\varphi_{\mathrm{4.5PN}} &= \Bigg(-\frac{93098188434443}{150214901760} + \frac{80}{3}\pi^2 + \frac{1712}{21}\gamma_{\mathrm{E}} + \frac{3424}{21}\ln 2 \\ &\qquad + \left[\frac{1492917260735}{1072963584} - \frac{2255}{48}\pi^2\right]\nu - \frac{45293335}{1016064}\nu^2 - \frac{10323755}{1596672}\nu^3\Bigg)\pi \\ \varphi^{(l)}_{\mathrm{4.5PN}} &= \frac{856}{21}\,\pi\end{aligned}}$$

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The 4.5PN radiative quadrupole moment

$$
\begin{aligned}
U_{ij}(t) = I^{(2)}_{ij}(t) &+ \underbrace{\frac{GM}{c^3}\int_0^{+\infty} \mathrm{d}\tau\, I^{(4)}_{ij}(t-\tau)\left[2\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{11}{6}\right]}_{\text{1.5PN tail integral}} \\
&+ \frac{G}{c^5}\left\{\underbrace{-\frac{2}{7}\int_0^{+\infty} \mathrm{d}\tau\, I^{(3)}_{a<i}I^{(3)}_{j>a}(t-\tau)}_{\text{2.5PN memory integral}} + \text{instantaneous terms}\right\} \\
&+ \underbrace{\frac{G^2M^2}{c^6}\int_0^{+\infty} \mathrm{d}\tau\, I^{(5)}_{ij}(t-\tau)\left[2\ln^2\left(\frac{\tau}{2\tau_0}\right)+\frac{57}{35}\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{124627}{22050}\right]}_{\text{3PN tail-of-tail integral}} \\
&+ \underbrace{\frac{G^3M^3}{c^9}\int_0^{+\infty} \mathrm{d}\tau\, I^{(6)}_{ij}(t-\tau)\left[\frac{4}{3}\ln^3\left(\frac{\tau}{2\tau_0}\right)+\cdots+\frac{129268}{33075}+\frac{428}{315}\pi^2\right]}_{\text{4.5PN tail-of-tail-of-tail integral}} \\
&+ \mathcal{O}\left(\frac{1}{c^{10}}\right)
\end{aligned}
$$

The 4PN mass type quadrupole moment

[Marchand, Henry, Larrouturou, Marsat, Faye & Blanchet, 2019 in progress]

1. The ℓ-th order mass type multipole moments of a general isolated source is

$$I_L(t) = \mathop{\mathrm{FP}}_{B=0} \int \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \int_{-1}^{1} \mathrm{d}z \Big\{ \delta_\ell \, \hat{x}_L \, \Sigma - \frac{4(2\ell+1)}{c^2(\ell+1)(2\ell+3)} \, \delta_{\ell+1} \, \hat{x}_{iL} \, \Sigma_i^{(1)} + \frac{2(2\ell+1)}{c^4(\ell+1)(\ell+2)(2\ell+5)} \, \delta_{\ell+2} \, \hat{x}_{ijL} \, \Sigma_{ij}^{(2)} \Big\}$$

2. Need components of the metric with 4PN accuracy

$$h^{00} = -\frac{4V}{c^2} - \frac{2}{c^4}\left(\hat{W} + 4V^2\right) - \frac{8}{c^6}\Big(\overbrace{\hat{X}}^{\text{2PN potential}} + \cdots \Big) - \frac{64}{c^8}\Big(\overbrace{\hat{T}}^{\text{3PN potential}} + \cdots \Big)$$

$$h^{0i} = \frac{4V_i}{c^3} + \frac{8}{c^5}\left(\hat{R}_i + V_i V\right) + \frac{16}{c^7}\Big(\overbrace{\hat{Y}_i}^{\text{3PN potential}} + \cdots \Big)$$

$$h^{ij} = -\frac{4}{c^4}\left(\hat{W}_{ij} - \frac{1}{2}\delta_{ij}\hat{W}\right) - \frac{16}{c^4}\Big(\underbrace{\hat{Z}_{ij}}_{\text{3PN potential}} - \frac{1}{2}\delta_{ij}\hat{Z}\Big) - \frac{32}{c^8}\Big(\underbrace{\hat{M}_{ij}}_{\text{4PN potential}} + \cdots \Big)$$

The 4PN mass type quadrupole moment

[Marchand, Henry, Larrouturou, Marsat, Faye & Blanchet, 2019 in progress]

1. Method of super-potentials [Blanchet, Faye & Whiting 2014]

$$\int \mathrm{d}^3\mathbf{x}\, r^B\, \hat{x}_L \overbrace{\phi}^{\text{linear potential}} \underbrace{P}_{\text{difficult potential}} = \int \mathrm{d}^3\mathbf{x}\, r^B \Big(\Psi^\phi_L\, \Delta P + \underbrace{\partial_i \Big[\partial_i \Psi^\phi_L P - \Psi^\phi_L \partial_i P\Big]\Big)}_{\text{yields a surface term}}$$

where Ψ^ϕ_L is obtained from the super-potentials ϕ_{2k} of $\phi = \phi_0$ as

$$\boxed{\Psi^\phi_L = \Delta^{-1}\big(\hat{x}_L\, \phi\big) = \sum_{k=0}^{\ell} \frac{(-2)^k \ell!}{(\ell-k)!}\, x_{\langle L-K}\partial_{K\rangle}\, \overbrace{\phi_{2k+2}}^{\Delta\phi_{2k+2} = \phi_{2k}}}$$

2. Method of surface integrals [Blanchet & Iyer 2005]

$$\boxed{\underset{B=0}{\mathrm{FP}} \int \mathrm{d}^3\mathbf{x}\, r^B \hat{x}_L\, \Delta G = -(2\ell+1) \int \mathrm{d}\Omega\, \hat{n}_L X_\ell(\boldsymbol{n})}$$

where X_ℓ is the coefficient of $r^{-\ell-1}$ in the expansion of G when $r \to +\infty$

The 4PN mass type quadrupole moment

[Marchand, Henry, Larrouturou, Marsat, Faye & Blanchet, 2019 in progress]

1. All terms admit a closed form expression and have been computed explicitly
2. Dimensional regularization is used to treat UV divergences
   - The UV poles are renormalized by means of the same UV shifts as determined in the 4PN equations of motion
3. IR divergences are treated for the moment using the Hadamard finite part when $B \to 0$ but will later be corrected by means of the B-ε regularization
   - The IR poles will be renormalized by means of the same IR shifts as found in the 4PN equations of motion
4. The 4PN radiative quadrupole moment is obtained by adding a specific coupling between the 1.5PN tail and the 2.5PN memory
5. The 4PN GW mode h_{22} follows directly from the radiative quadrupole
6. Adding the contributions of the 3PN mass octupole and 3PN current quadrupole moments we compute the 4PN GW flux and phase evolution

RADIATION REACTION AND FLUX-BALANCE EQUATIONS

Radiation reaction to 4PN order

1. At 2.5PN order for general matter systems the radiation reaction force in a specific gauge is purely scalar [Burke & Thorne 1970]

$$F_i^{\text{reac}} = \rho\,\partial_i V^{\text{reac}}$$

2. At the 3.5PN order the radiation reaction derives from scalar and vector radiation reaction potentials

$$F_i^{\text{reac}} = \rho\left[\partial_i V^{\text{reac}} - \frac{4}{c^2}v^j\left(\partial_i V_j^{\text{reac}} - \partial_j V_i^{\text{reac}}\right) - \frac{4}{c^3}\varepsilon_{ijk}v^j\frac{\mathrm{d}V_k^{\text{reac}}}{\mathrm{d}t}\right]$$

3. At 4PN order the radiation reaction contains a tail term (again scalar)

Radiation reaction to 4PN order [Blanchet 1993, 1997]

$$V^{\text{reac}} = - \underbrace{\frac{G}{5c^5} x^{ij} I^{(5)}_{ij}}^{\text{2.5PN radiation reaction}} + \overbrace{\frac{G}{c^7}\left[\frac{1}{189} x^{ijk} I^{(7)}_{ijk} - \frac{1}{70} r^2 x^{ij} I^{(7)}_{ij}\right]}^{\text{3.5PN scalar correction}}$$
$$\underbrace{- \frac{4G^2M}{5c^8} x^{ij} \int_0^{+\infty} \mathrm{d}\tau\, I^{(7)}_{ij}(t-\tau) \left[\ln\left(\frac{\tau}{2\tau_0}\right) + \frac{11}{12}\right]}_{\text{4PN radiation reaction tail}} + \mathcal{O}\left(\frac{1}{c^9}\right)$$
$$V^{\text{reac}}_i = \underbrace{\frac{G}{c^5}\left[\frac{1}{21} \hat{x}^{ijk} I^{(6)}_{jk} - \frac{4}{45} \varepsilon_{ijk}\, x^{jl} J^{(5)}_{kl}\right]}_{\text{3.5PN vector correction}} + \mathcal{O}\left(\frac{1}{c^7}\right)$$

This result permits to prove the balance equations for general isolated systems up to the 4PN order or 1.5PN relative order beyond the quadrupolar radiation

Radiation reaction derivation revisited [Blanchet & Faye 2018]

1. Metric accurate to 1PN order for conservative effects and to 3.5PN order for dissipative radiation reaction effects

$$
\begin{aligned}
g_{00} &= -1 + \frac{2\mathcal{V}}{c^2} - \frac{2\mathcal{V}^2}{c^4} + \mathcal{O}^{\text{cons}}\left(\frac{1}{c^6}\right)\\
g_{0i} &= -\frac{4\mathcal{V}_i}{c^3} + \mathcal{O}^{\text{cons}}\left(\frac{1}{c^5}\right)\\
g_{ij} &= \delta_{ij}\left(1 + \frac{2\mathcal{V}}{c^2}\right) + \frac{4}{c^4}\Big(W_{ij} - \delta_{ij}W_{kk}\Big) + \mathcal{O}^{\text{cons}}\left(\frac{1}{c^6}\right)
\end{aligned}
$$

2. Potentials are composed of a conservative part and a dissipative one

$$\mathcal{V}_\mu = V_\mu^{\text{cons}} + \boxed{V_\mu^{\text{reac}}}$$

3. Integrate the matter equations of motion $\nabla_\nu T^{\mu\nu} = 0$ over the source

$$\partial_\nu\left(\sqrt{-g}T^\nu_\mu\right) = \frac{1}{2}\sqrt{-g}\,\partial_\mu g_{\rho\sigma}T^{\rho\sigma}$$

Radiation reaction derivation revisited [Blanchet & Faye 2018]

- Recover well known results for the fluxes of energy and angular momentum [Epstein & Wagoner 1975; Thorne 1980; Blanchet & Damour 1989]

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{G}{c^5}\left(\frac{1}{5}I^{(3)}_{ij}I^{(3)}_{ij} + \frac{1}{c^2}\left[\frac{1}{189}I^{(4)}_{ijk}I^{(4)}_{ijk} + \frac{16}{45}J^{(3)}_{ij}J^{(3)}_{ij}\right]\right) + \mathcal{O}\left(\frac{1}{c^8}\right)$$

$$\frac{\mathrm{d}J_i}{\mathrm{d}t} = -\frac{G}{c^5}\varepsilon_{ijk}\left(\frac{2}{5}I^{(2)}_{jl}I^{(3)}_{kl} + \frac{1}{c^2}\left[\frac{1}{63}I^{(3)}_{jlm}I^{(4)}_{klm} + \frac{32}{45}J^{(2)}_{jl}J^{(3)}_{kl}\right]\right) + \mathcal{O}\left(\frac{1}{c^8}\right)$$

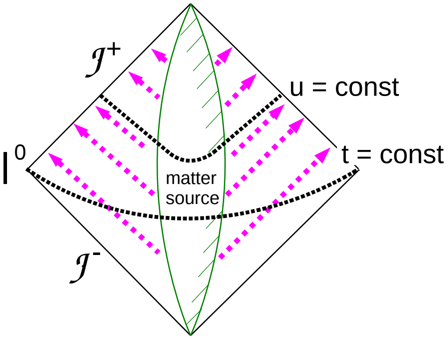
- And also for the linear momentum which is a subdominant 3.5PN effect [Papapetrou 1971; Bekenstein 1973]

$$\frac{\mathrm{d}P_i}{\mathrm{d}t} = -\frac{G}{c^7}\left[\frac{2}{63}I^{(4)}_{ijk}I^{(3)}_{jk} + \frac{16}{45}\varepsilon_{ijk}I^{(3)}_{jl}J^{(3)}_{kl}\right] + \mathcal{O}\left(\frac{1}{c^9}\right)$$

What about the position of the center of mass?

1. For an isolated conservative system the conserved integrals are E, J_i, P_i and also the initial position of the center of mass
$$Z_i = G_i - P_i t$$
where G_i is the position of the center of mass multiplied by the mass
2. The conservation of Z_i is associated with the invariance under Lorentz boosts
3. We also find a balance equation for the center-of-mass position
$$\boxed{\frac{\mathrm{d}G_i}{\mathrm{d}t} = P_i - \frac{2G}{21c^7}\, I^{(3)}_{ijk} I^{(3)}_{jk} + \mathcal{O}\left(\frac{1}{c^9}\right)}$$
4. This formula has never appeared in standard texbooks on GR or gravitational waves, nor on specialized reviews, it appeared only recently in the GW litterature [Kozameh, Nieva & Quirega 2018; Blanchet & Faye 2018; Compère, Oliveri & Seraj 2019]

Direct calculation of the GW fluxes at infinity



1. Introduce a retarded null coordinate u satisfying

$$\boxed{g^{\mu\nu}\partial_\mu u \partial_\nu u = 0}$$

2. For instance choose $u = t - r_*/c$ with the tortoise coordinate

$$r_* = r + \frac{2GM}{c^2}\ln\left(\frac{r}{r_0}\right) + \mathcal{O}\left(\frac{1}{r}\right)$$

Direct calculation of the GW fluxes at infinity

1. Perform a coordinate change $(t, \mathbf{x}) \rightarrow (u, \mathbf{x})$ in the conservation law of the pseudo-tensor $\partial_\nu \tau^{\mu\nu} = 0$ to get

$$\frac{\partial}{c\partial u}\Big[\tau^{\mu 0}(\mathbf{x}, u + r_*/c) - n_*^i \tau^{\mu i}(\mathbf{x}, u + r_*/c)\Big] + \partial_i\Big[\tau^{\mu i}(\mathbf{x}, u + r_*/c)\Big] = 0$$

2. Integrating over a volume $\mathcal{V}$ tending to infinity with $u =$const

$$\boxed{\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}u} &= -c \int_{\partial\mathcal{V}} \mathrm{d}S_i \, \tau^{0i}_{\mathsf{GW}}(\mathbf{x}, u + r_*/c) \\
\frac{\mathrm{d}J_i}{\mathrm{d}u} &= -\varepsilon_{ijk} \int_{\partial\mathcal{V}} \mathrm{d}S_l \, x^j \, \tau^{kl}_{\mathsf{GW}}(\mathbf{x}, u + r_*/c) \\
\frac{\mathrm{d}P^i}{\mathrm{d}u} &= - \int_{\partial\mathcal{V}} \mathrm{d}S_j \, \tau^{ij}_{\mathsf{GW}}(\mathbf{x}, u + r_*/c) \\
\frac{\mathrm{d}G_i}{\mathrm{d}u} &= P_i - \frac{1}{c} \int_{\partial\mathcal{V}} \mathrm{d}S_j \left(x^i \, \tau^{0j}_{\mathsf{GW}} - r_* \, \tau^{ij}_{\mathsf{GW}} \right) (\mathbf{x}, u + r_*/c)
\end{aligned}}$$

Direct calculation of the GW fluxes at infinity

A long calculation to control the leading $1/r^2$ and subleading $1/r^3$ terms in the GW pseudo-tensor when $r \to +\infty$ gives the fluxes as full multipole series parametrized by the multipole moments I_L and J_L up to order $\mathcal{O}(G^2)$

$$\boxed{\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}u} &= -\sum_{\ell=2}^{+\infty} \frac{G}{c^{2\ell+1}} \left\{ \frac{(\ell+1)(\ell+2)}{(\ell-1)\ell\ell!(2\ell+1)!!} \overset{(\ell+1)}{I}_L \overset{(\ell+1)}{I}_L \right. \\
&\qquad\qquad \left. + \frac{4\ell(\ell+2)}{c^2(\ell-1)(\ell+1)!(2\ell+1)!!} \overset{(\ell+1)}{J}_L \overset{(\ell+1)}{J}_L \right\} \\
\frac{\mathrm{d}J_i}{\mathrm{d}u} &= -\varepsilon_{ijk} \sum_{\ell=2}^{+\infty} \frac{G}{c^{2\ell+1}} \left\{ \frac{(\ell+1)(\ell+2)}{(\ell-1)\ell!(2\ell+1)!!} \overset{(\ell)}{I}_{jL-1} \overset{(\ell+1)}{I}_{kL-1} \right. \\
&\qquad\qquad \left. + \frac{4\ell^2(\ell+2)}{c^2(\ell-1)(\ell+1)!(2\ell+1)!!} \overset{(\ell)}{J}_{jL-1} \overset{(\ell+1)}{J}_{kL-1} \right\}
\end{aligned}}$$

Direct calculation of the GW fluxes at infinity

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$$\boxed{\begin{aligned}
\frac{\mathrm{d}P_i}{\mathrm{d}u} &= -\sum_{\ell=2}^{+\infty} \frac{G}{c^{2\ell+3}} \Bigg\{ \frac{2(\ell+2)(\ell+3)}{\ell(\ell+1)!(2\ell+3)!!} \overset{(\ell+2)}{I}_{iL} \overset{(\ell+1)}{I}_{L} \\
&\qquad + \frac{8(\ell+2)}{(\ell-1)(\ell+1)!(2\ell+1)!!} \varepsilon_{ijk} \overset{(\ell+1)}{I}_{jL-1} \overset{(\ell+1)}{J}_{kL-1} \\
&\qquad + \frac{8(\ell+3)}{c^2(\ell+1)!(2\ell+3)!!} \overset{(\ell+2)}{J}_{iL} \overset{(\ell+1)}{J}_{L} \Bigg\} \\
\frac{\mathrm{d}G_i}{\mathrm{d}u} &= P_i \\
&\quad \underbrace{- \sum_{\ell=2}^{+\infty} \frac{G}{c^{2\ell+3}} \Bigg\{ \frac{2(\ell+2)(\ell+3)}{\ell\,\ell!(2\ell+3)!!} \overset{(\ell+1)}{I}_{iL} \overset{(\ell+1)}{I}_{L} + \frac{8(\ell+3)}{c^2\ell!(2\ell+3)!!} \overset{(\ell+1)}{J}_{iL} \overset{(\ell+1)}{J}_{L} \Bigg\}}_{\text{[Blanchet \& Faye 2018; Compère, Oliveri \& Seraj 2019]}}
\end{aligned}}$$

Any implication for the total recoil of a source?

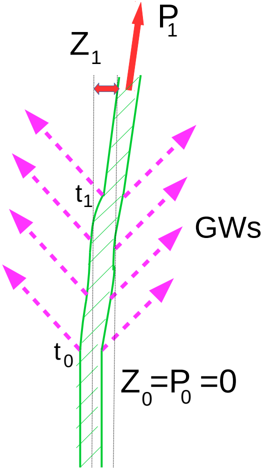
1. We have obtained the balance equations

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = -\boldsymbol{F}_P\,,$$
$$\frac{\mathrm{d}\boldsymbol{G}}{\mathrm{d}t} = \boldsymbol{P} - \boldsymbol{F}_G\,,$$

2. Integrating these equations for a burst of GWs with finite duration we obtain

$$\boldsymbol{P}_1 = -\int_{t_0}^{t_1} \mathrm{d}t'\,\boldsymbol{F}_P(t')\,,$$
$$\boldsymbol{Z}_1 = \int_{t_0}^{t_1} \mathrm{d}t' \Big[t'\,\boldsymbol{F}_P(t') - \boldsymbol{F}_G(t')\Big]\,.$$

3. The total recoil depends only on the linear momentum flux (as in usual calculations)



The instantaneous CM position of a circular binary

1. The linear momentum is evaluated for a Newtonian circular binary as usual [Fitchett 1983]

$$\boxed{\begin{aligned}\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} &= \frac{464}{105}\frac{G^4 m^5 \omega}{c^7 r^4}\sqrt{1-4\nu}\,\nu^2\,\boldsymbol{\lambda}\\ \boldsymbol{P} &= \frac{464}{105}\frac{G^4 m^5}{c^7 r^4}\sqrt{1-4\nu}\,\nu^2\,\boldsymbol{n}\end{aligned}}$$

2. However in order to obtain the instantaneous CM position we must also use the CM flux

$$\boxed{\begin{aligned}\frac{\mathrm{d}\boldsymbol{G}}{\mathrm{d}t} &= \boldsymbol{P} + \frac{544}{105}\frac{G^4 m^5}{c^7 r^4}\sqrt{1-4\nu}\,\nu^2\,\boldsymbol{n}\\ \boldsymbol{G} &= -\frac{48}{5}\frac{G^4 m^5}{c^7 r^4 \omega}\sqrt{1-4\nu}\,\nu^2\,\boldsymbol{\lambda}\end{aligned}}$$

3. It would be interesting to compare this prediction to very accurate NR computations of the CM position [Gerosa, Hébert & Stein 2018; Woodford, Boyle & Pfeiffer 2019]