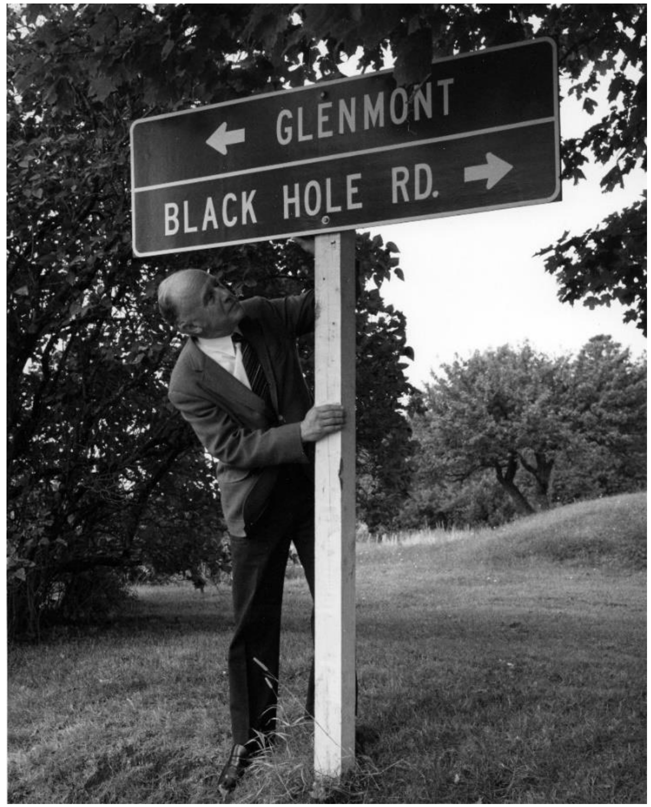
Black holes as particle detectors



Vitor Cardoso (Lisbon)

1919. May 29 eclips confirms that gravity “bends” light

REVOLUTION IN SCIENCE.

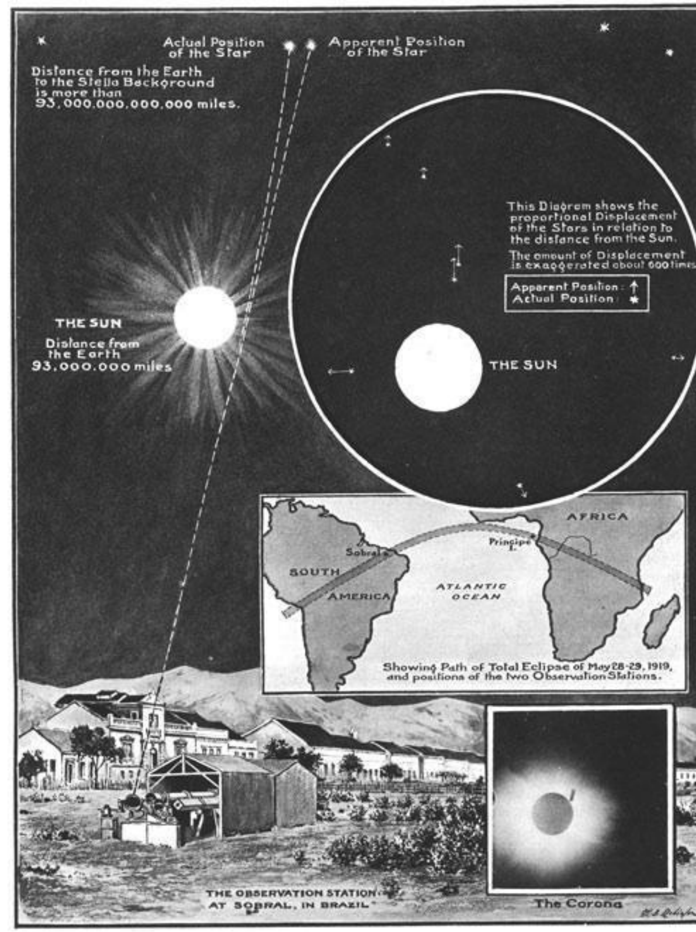
NEW THEORY OF THE UNIVERSE.

NEWTONIAN IDEAS OVERTHROWN.

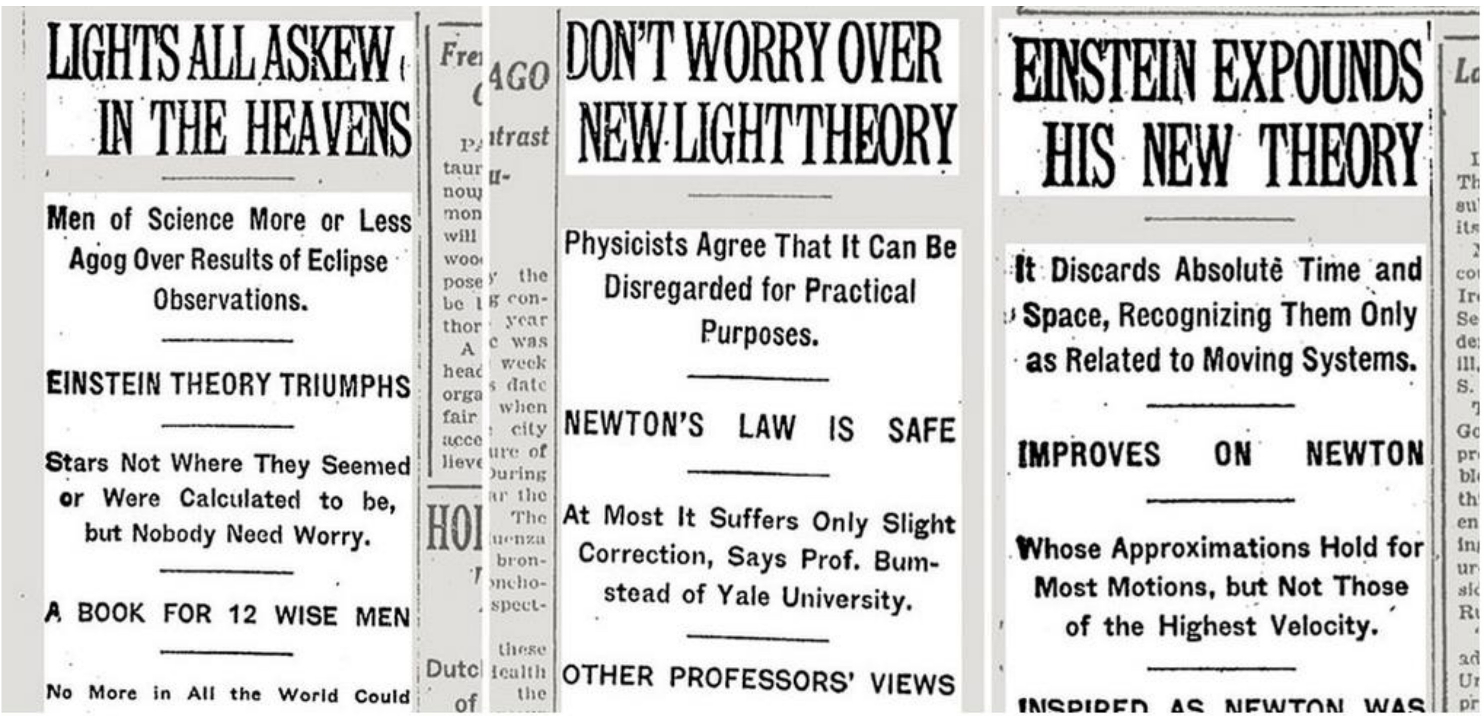
Yesterday afternoon in the rooms of the Royal Society, at a joint session of the Royal and Astronomical Societies, the results obtained by British observers of the total solar eclipse of May 29 were discussed.

The greatest possible interest had been aroused in scientific circles by the hope that rival theories of a fundamental physical problem would be put to the test, and there was a very large attendance of astronomers and physicists. It was generally accepted that the observations were decisive in the verifying of the prediction of the famous physicist, Einstein, stated by the President of the Royal Society as being the most remarkable scientific event since the discovery of the predicted existence of the planet Neptune. But there was differ-

‘Times of London‘, Nov 7 1919



'Illustrated London News‘, Nov 22 1919



LIGHTS ALL ASKEW IN THE HEAVENS

Men of Science More or Less Agog Over Results of Eclipse Observations.

EINSTEIN THEORY TRIUMPHS

Stars Not Where They Seemed or Were Calculated to be, but Nobody Need Worry.

A BOOK FOR 12 WISE MEN

No More in All the World Could

DON'T WORRY OVER NEW LIGHT THEORY

Physicists Agree That It Can Be Disregarded for Practical Purposes.

NEWTON'S LAW IS SAFE

At Most It Suffers Only Slight Correction, Says Prof. Bumstead of Yale University.

OTHER PROFESSORS' VIEWS

EINSTEIN EXPOUNDS HIS NEW THEORY

It Discards Absolute Time and Space, Recognizing Them Only as Related to Moving Systems.

IMPROVES ON NEWTON

Whose Approximations Hold for Most Motions, but Not Those of the Highest Velocity.

INSPIRED AS NEWTON WAS

Articles in The Times from Nov. 10, 1919, left; Nov. 16, 1919, center; and Dec. 3, 1919.

Uniqueness: the Kerr solution

Theorem (Carter 1971; Robinson 1975; Chrusciel and Costa 2012):
A stationary, asymptotically flat, vacuum BH solution must be Kerr

$$
\begin{aligned}
ds^2 \;&=\; \frac{\Delta - a^2\sin^2\theta}{\Sigma}dt^2 + \frac{2a(r^2+a^2-\Delta)\sin^2\theta}{\Sigma}dtd\phi \\
&\quad - \frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta d\phi^2 - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2
\end{aligned}
$$

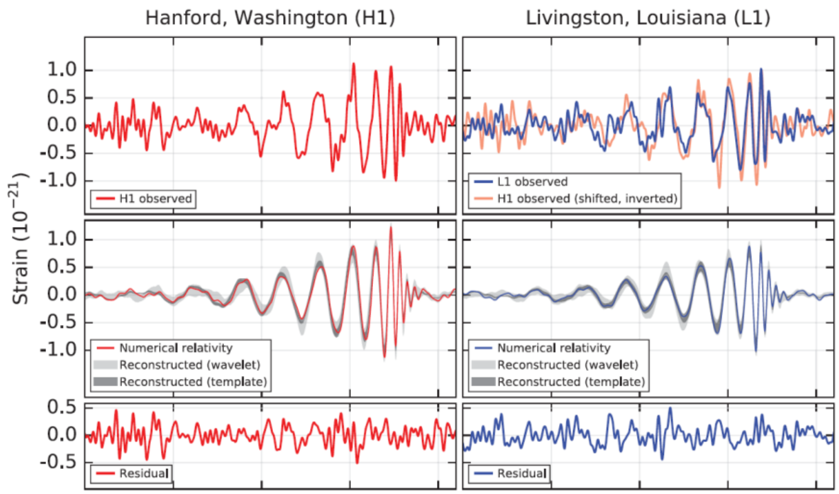
$$
\Sigma = r^2 + a^2\cos^2\theta\,, \quad \Delta = r^2 + a^2 - 2Mr
$$

Describes a rotating BH with mass M and angular momentum J=aM, iff a<M

"In my entire scientific life, extending over forty-five years, the most shattering experience has been the realization that an exact solution of Einstein's equations of general relativity provides the *absolutely exact representation* of untold numbers of black holes that populate the universe."

S. Chandrasekhar, The Nora and Edward Ryerson lecture, Chicago April 22 1975

Black holes exist



Abbott + PRL.116:061102 (2016)

BH seeds, demography… (how many, where, how?)

See review Barack+ arXiv:1806.05195

What is graviton mass or speed?

See review Barack+ arXiv:1806.05195

Are there extra radiation channels, corrections to gravity?

Barack+arXiv:1806.05195; Barausse+PRL116:241104(2016);

Is the final - or initial - object really a black hole?

Cardoso+ PRL116: 171101 (2016); Cardoso & Pani, Nature Astronomy 1: 586 (2017)

Is it a Kerr black hole? Can we constrain alternatives?

Berti+ 2005, 2016; Cardoso & Gualtieri 2016

Can GWs from BHs inform us on fundamental fields/DM?

Barack+arXiv:1806.05195; Arvanitaki+ PRD95: 043001 (2016); Brito+ PRL119:131101 (2017)

Precision physics: the inspiral phase

$$h_{+,\times} = \frac{2Gm\eta}{c_0^2 r}\left(\frac{Gm\omega}{c_0^3}\right)^{2/3}\left\{H^{(0)}_{+,\times} + x^{1/2}H^{(1/2)}_{+,\times} + xH^{(1)}_{+,\times} + x^{3/2}H^{(3/2)}_{+,\times} + x^2 H^{(2)}_{+,\times}\right\}$$

G is Newton's constant, c_0 is speed of light

m is total mass, η is chirp mass, r is distance to source

ω is orbital frequency

x is velocity

c,s are cos and sin of inclination angle

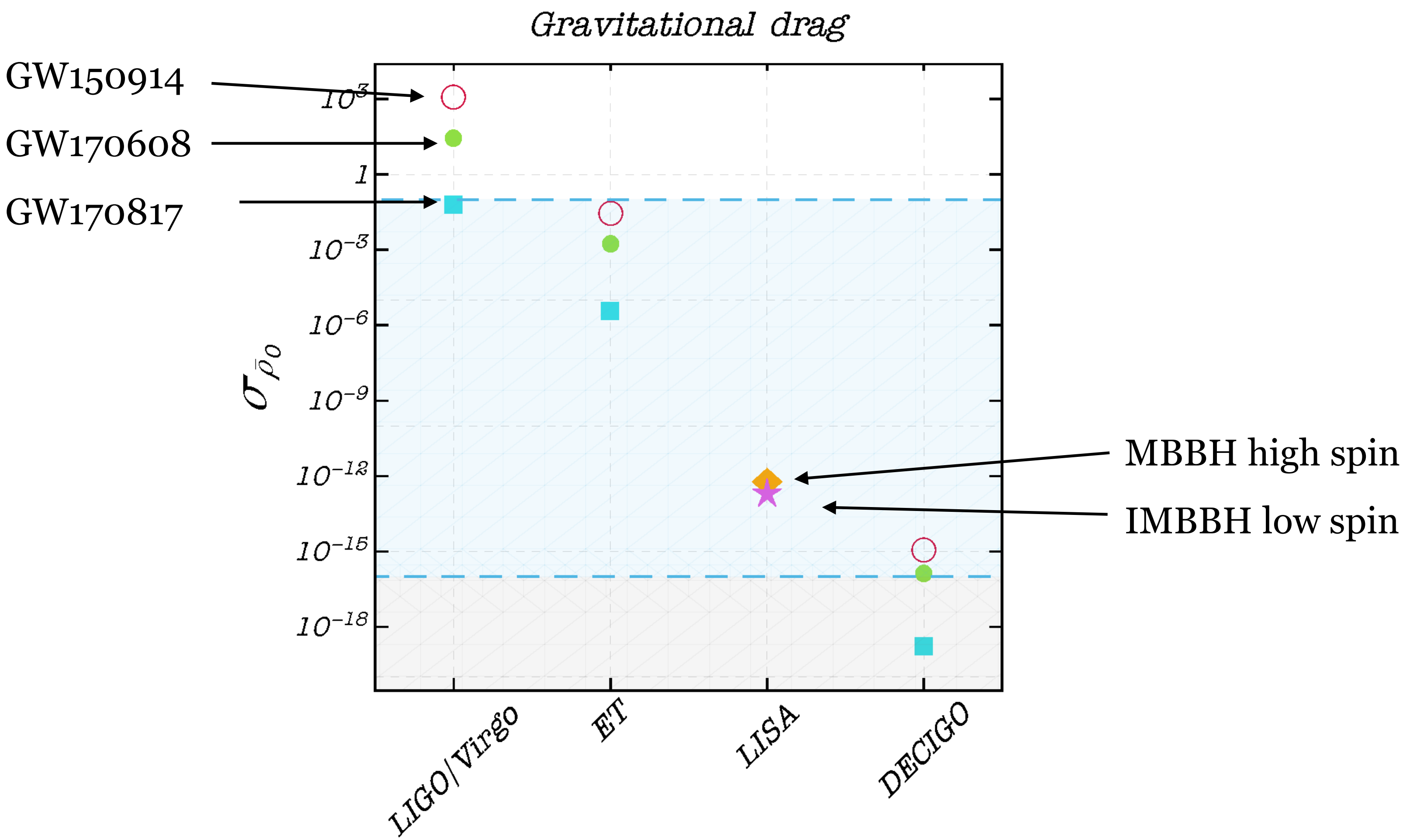
ψ is, up to a constant, the orbital phase

Θ= η/(5m)(to-t)

$$\phi(t) = \phi_c - \frac{1}{\eta}\left\{\Theta^{5/8} + \left(\frac{3715}{8064} + \frac{55}{96}\eta\right)\Theta^{3/8} - \frac{3\pi}{4}\Theta^{1/4} + \left(\frac{9275495}{14450688} + \frac{284875}{258048}\eta + \frac{1855}{2048}\eta^2\right)\Theta^{1/8}\right\}$$

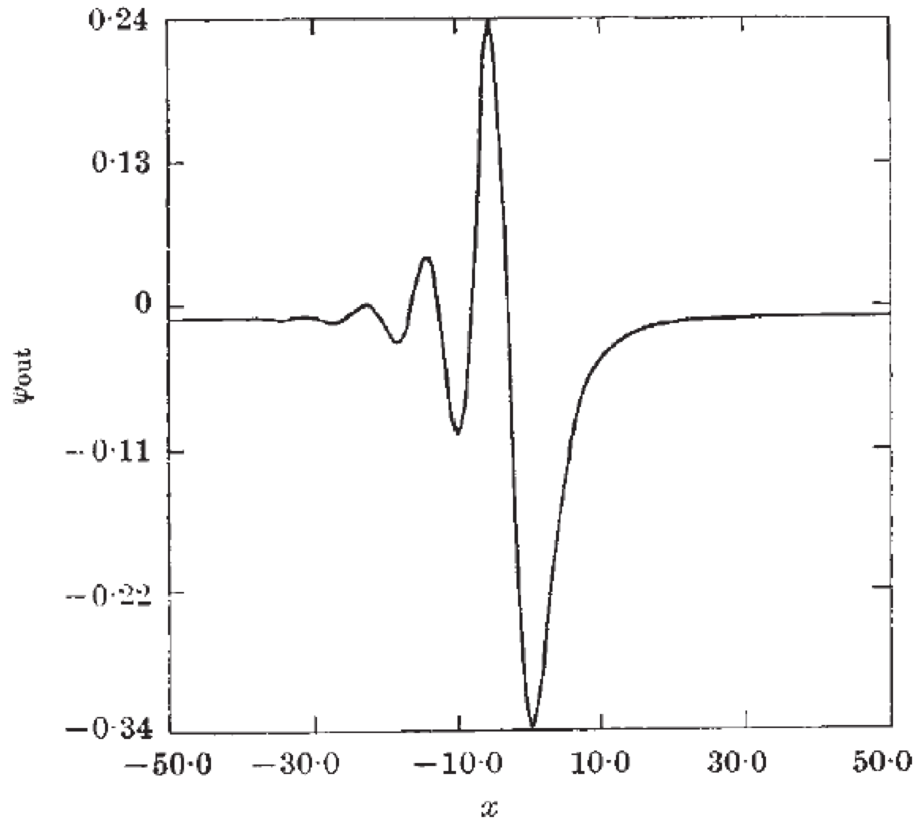
Blanchet, Iyer, Will, Wiseman, *CQG13:575 (1996)*

Precision physics: testing environments

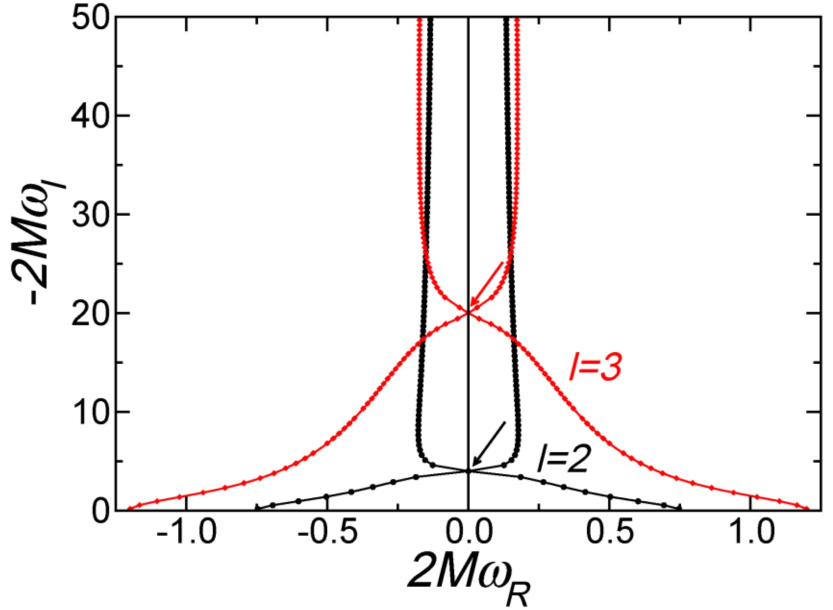


Cardoso & Maselli arXiv 1909.05870; Barausse + PRD89 (2014) 104059
Also Eda + PRL 110 (2013) 221101; Macedo+ApJ774 (2013) 48

Precision physics with hair loss: the modes of black holes



C.V.Vishveshwara, Nature 227: 938 (1970)
Data and routines at blackholes.ist.utl.pt



$$f = \omega_R/2\pi = 1.207\left(\frac{10\ M_\odot}{M}\right)\text{kHz}$$

$$\tau = 1/|\omega_I| = 0.5537\left(\frac{M}{10\ M_\odot}\right)\text{ms}$$

Berti+ PRD73: 064030 (2006); Berti + CQG 26: 163001 (2009)

"*After the advent of gravitational wave astronomy, the observation of these resonant frequencies might finally provide direct evidence of BHs with the same certainty as, say, the 21 cm line identifies interstellar hydrogen*" (S. Detweiler ApJ239:292 1980)

New fundamental fields and couplings

Can we catalogue all theories? When no additional fields, EFT suggests that generic (local) theories are described by EFT action Endlich+ JHEP1709:122

$$S_{\rm eff} = \int d^4x\sqrt{-g}\,2M_{\rm pl}^2\left(R - \frac{\mathcal{C}^2}{\Lambda^6} - \frac{\tilde{\mathcal{C}}^2}{\tilde{\Lambda}^6} - \frac{\tilde{\mathcal{C}}\mathcal{C}}{\Lambda_-^6}\right)$$

$$\mathcal{C} \equiv R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\,, \quad \tilde{\mathcal{C}} \equiv R_{\alpha\beta\gamma\delta}\tilde{R}^{\alpha\beta\gamma\delta}\,,$$

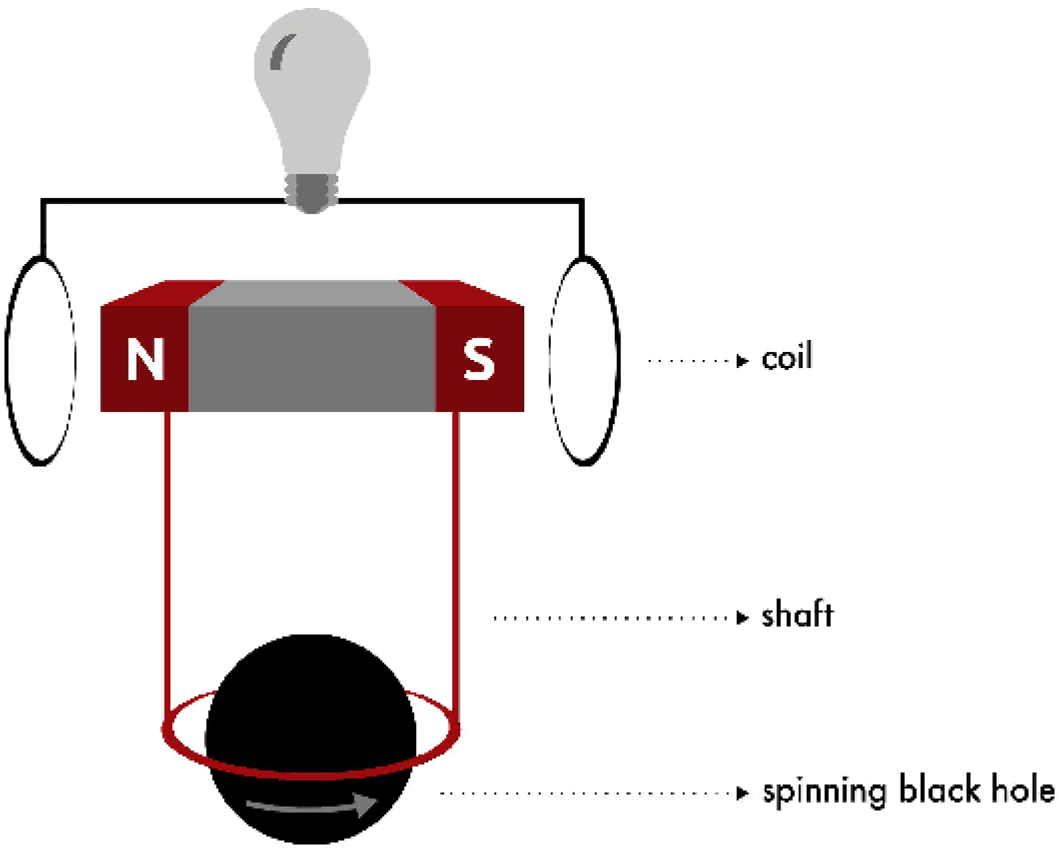
Systematic way to search for imprints with precision physics: compute BH solutions, explore dynamics Cardoso+ PRL121 (2019)251105

Are GR effects and degeneracies sufficiently understood (spin, spin-precession, eccentricity)? Are tidal Love numbers measurable when small?

Can we catalogue all possible ringdown frequencies, including theories where extra fields are present?

Cardoso+ PRD99 (2019) 104077; Tattersall + PRD 99 (2019) 104082; arXiv:1911.07593

Energy source?

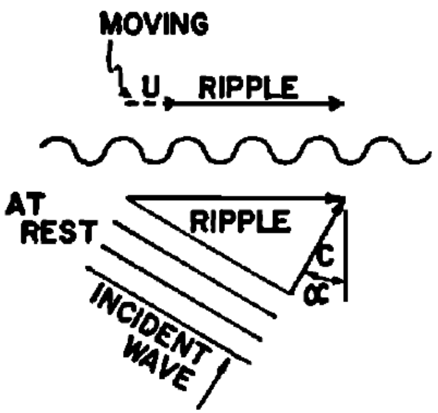


Penrose, Gravitational Collapse: the role of General Relativity (1969)

Brito, Cardoso & Pani, *Superradiance* (Springer-Verlag 2015)

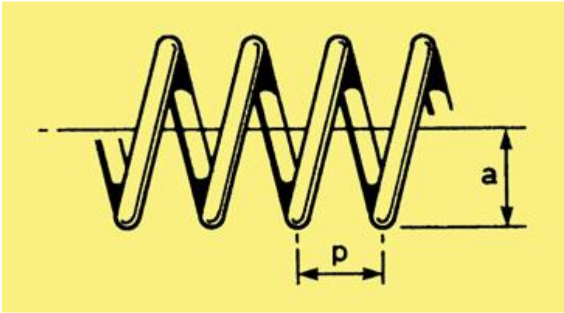
Superradiance

Zel'dovich JETP Lett. 14:180 (1971); Brito+ Lect. Notes Phys.906 (2015)

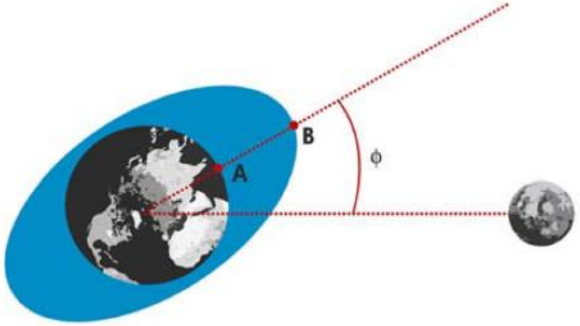


Ribner, J. Acous. Soc. Amer.29 (1957)

Tamm & Frank, Doklady AN SSSR 14 (1937)

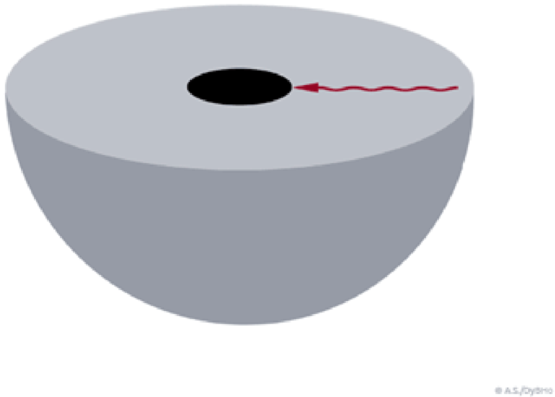


Pierce (& Kompfner), Bell Lab Series (1947)
Ginzburg, anomalous Doppler year



G. H. Darwin, Philos. Trans.
R. Soc. London 171 (1880)

Bombs and superradiant instabilities



$$\nabla_\gamma \nabla^\gamma \Psi = \mu^2 \Psi\,, \quad \nabla_\gamma F^{\gamma\nu} = \mu^2 A^\nu\,, \quad \nabla_\gamma \nabla^\gamma h_{\mu\nu} = \mu^2 h_{\mu\nu}$$

$$\Psi \sim e^{-i\omega t} Y_{lm}$$

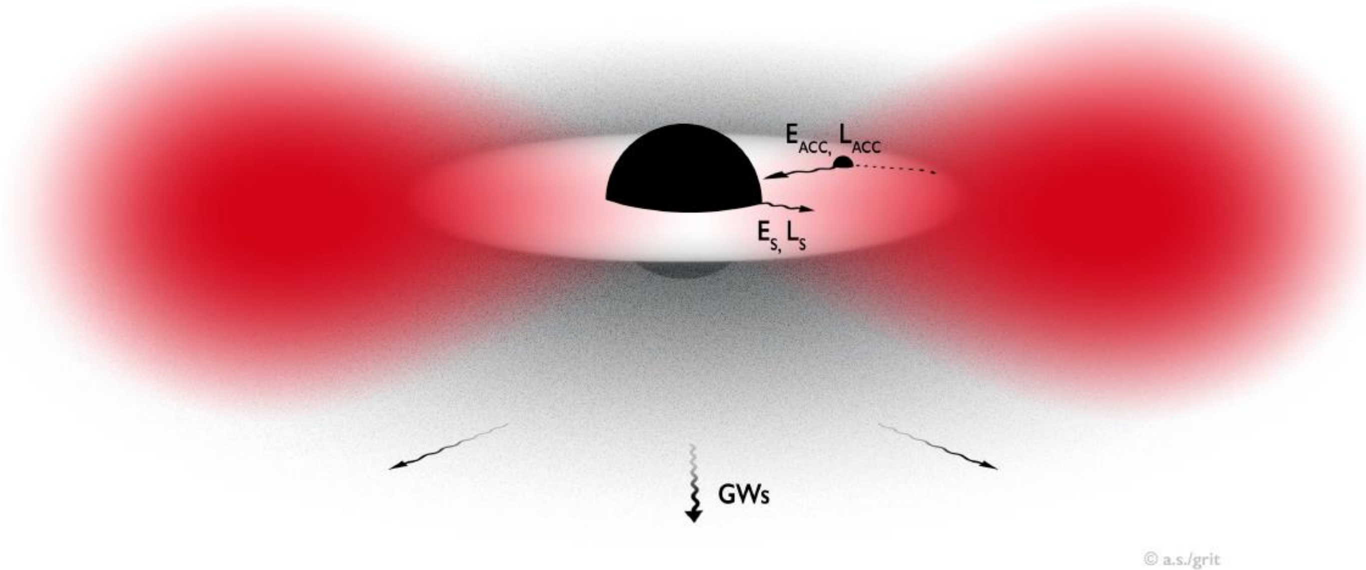
$$\omega \sim \mu + i(m\Omega_H - \mu)(M\mu)^{4l+5+S}$$

$$S = -s, -s+1 \ldots, s-1, s$$

Massive "states" around Kerr are linearly unstable

See review Brito+ Lect. Notes Phys. 906: 1 (2015, edition 2 in preparation)

Fundamental fields: bounding the boson mass

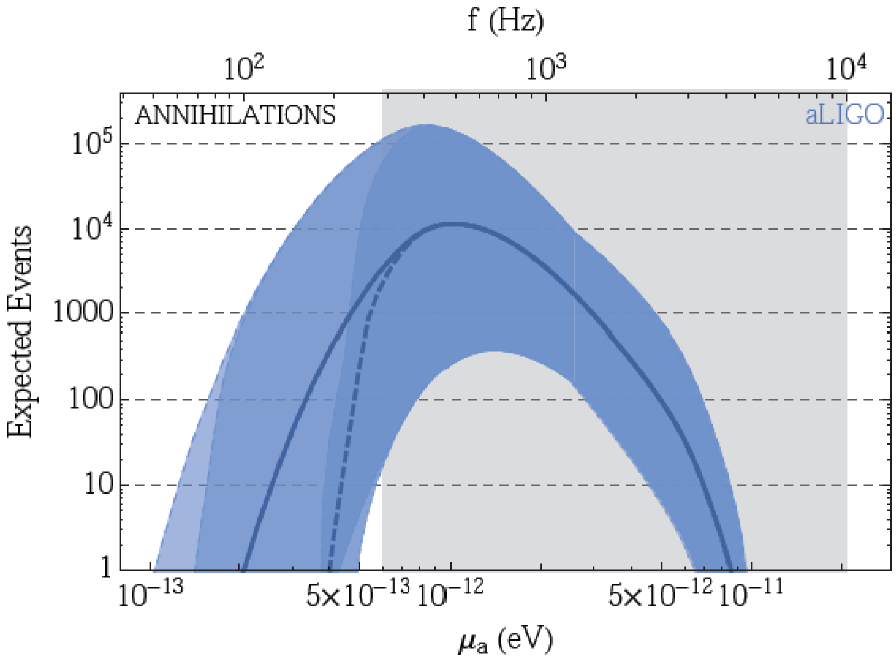


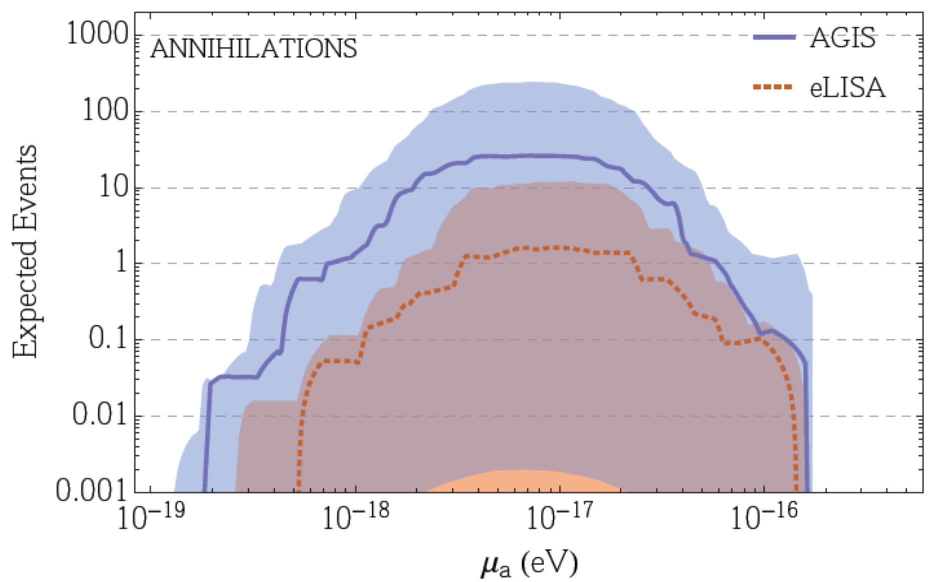
$$\tau \sim 100 \left(\frac{10^6 M_\odot}{M} \right)^8 \left(\frac{10^{-16}\mathrm{eV}}{\mu} \right)^9 \text{ seconds}$$

Wonderful sources of GWs

Brito+ Lecture Notes Physics 906: 1-237 (2015)

Wonderful sources for different GW-detectors!





Arvanitaki+ PRD91:084011 (2015);Brito+CQG32:134001 (2015);

Brito+ Lect.Notes Physics 906 (2015)

Wonderful sources for different GW-detectors!

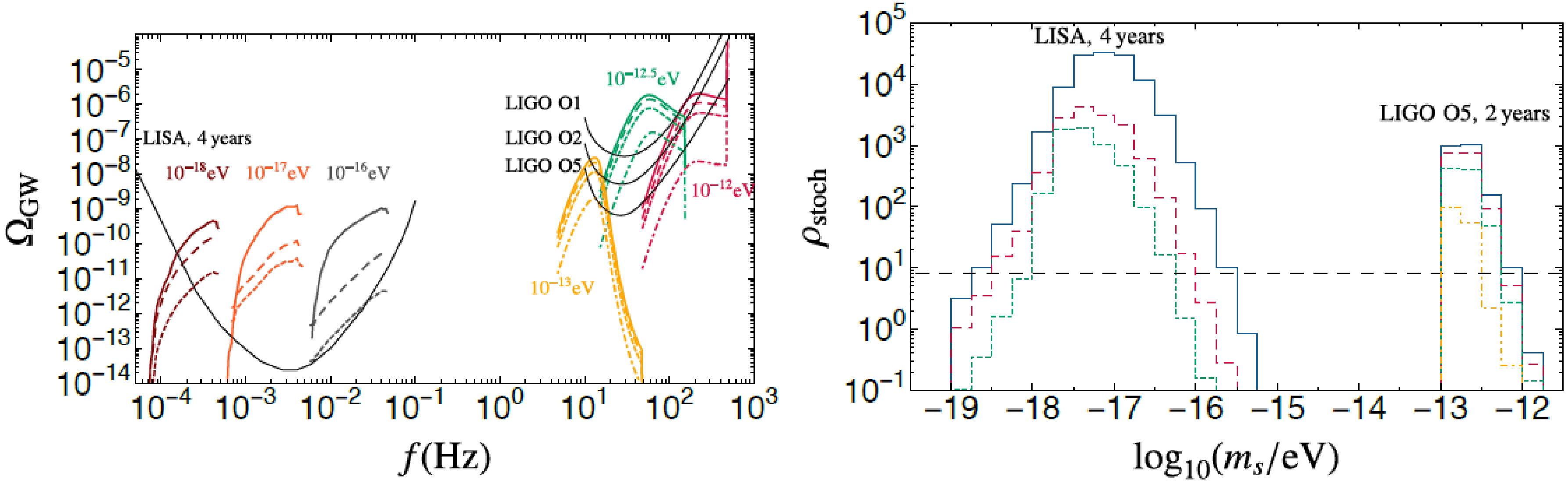
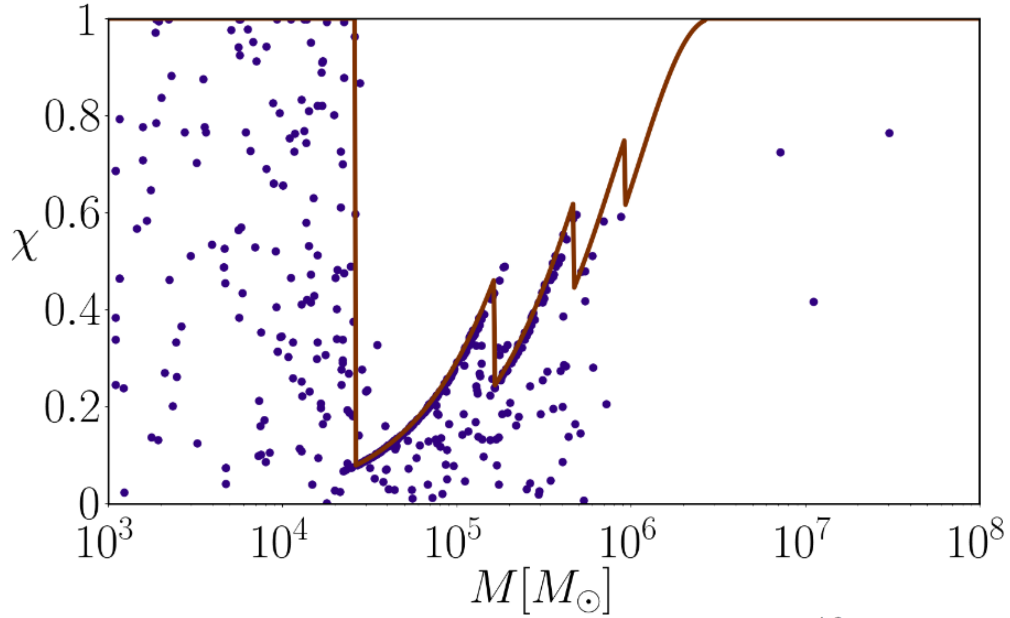


FIG. 2. Left panel: stochastic background in the LIGO and LISA bands. For LISA, the three different signals correspond to the "optimistic" (top), "less optimistic" (middle) and "pessimistic" (bottom) astrophysical models. For LIGO, the different spectra for each scalar field mass correspond to a uniform spin distribution with (from top to bottom) $\chi_i \in [0.8, 1]$, $[0.5, 1]$, $[0, 1]$ and $[0, 0.5]$. The black lines are the power-law integrated curves of Ref. [61], computed using noise PSDs for LISA [9], LIGO's first two observing runs (O1 and O2), and LIGO at design sensitivity (O5) [62]. By definition, $\rho_{\rm stoch} \geq 1$ when a power-law spectrum intersects one of the power-law integrated curves. Right panel: $\rho_{\rm stoch}$ for the backgrounds shown in the left panel. We assumed $T_{\rm obs} = 2\,\mathrm{yr}$ for LIGO and $T_{\rm obs} = 4\,\mathrm{yr}$ for LISA.

Brito + PRL119: 131101 (2017); arXiv: 1706:05097

PRD96:064050 (2017); arXiv: 1706.06311

Signatures in Regge plane



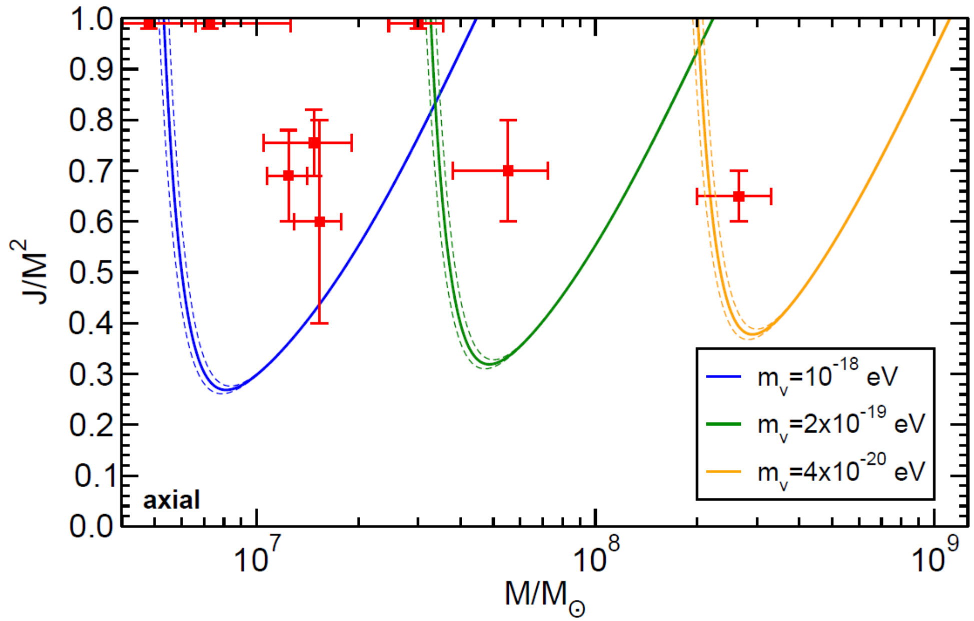
Two-year simulation for LISA and a boson with $10^{-16}\,\mathrm{eV}$. Saw-tooth due to different m harmonics. Final estimate from LISA: $(0.88-1.35)\times10^{-16}\,\mathrm{eV}$

Brito + PRL119: 131101 (2017); arXiv: 1706:05097

PRD96:064050 (2017); arXiv: 1706.06311

Bounding the boson mass with EM observations

Pani et al PRL109, 131102 (2012)



Bound on photon mass is model-dependent: details of accretion disks or intergalactic matter are important… but gravitons interact very weakly!

$$m_g < 5 \times 10^{-23}\,\mathrm{eV}$$

Brito + PRD88:023514 (2013); Review of Particle Physics 2014

Constraints on fundamental fields via superradiance

Review in Brito+ Lect. Notes Phys.906 (2015) (to be updated soon)

Resolvable events from single sources

Arvanitaki+ PRD91 (2015) 084011; Brito CQG32 (2015)134001; Brito+PRD96:064050; D'Antonio+PRD98 (2018)103017; Isi+ PRD99 (2019)084042; Palomba+ PRL123 (2019) 171101

Stochastic background

Brito+PRL119: 131101 (2017); PRD96:064050 (2017); Tsukada+PRD99 (2019) 103015; LLIGO/Virgo PRD100 (2019) 061101

Accurate measurements of BH spin (via EM or GW measurements)

Pani+ PRL 109 (2012) 131102; Brito+PRD88 (2013) 023514

Spin distribution

Arvanitaki+ PRD83 (2011) 044026; Brito CQG32 (2015)134001; Brito+PRL119: 131101 (2017); PRD96:064050 (2017);

Polarization of light if field is axionlike

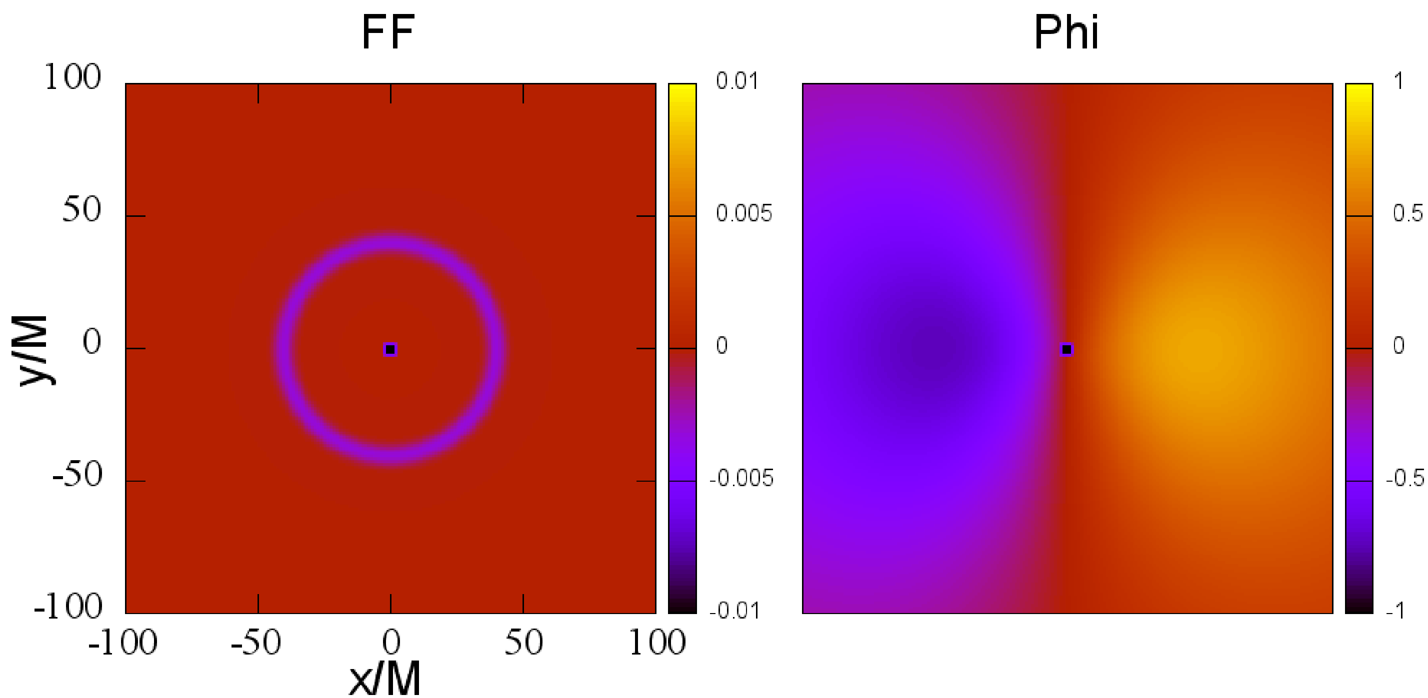
Plascencia+JCAP1804 (2018) 059; Chen+ arXiv:1905.02213

Motion of stars close to supermassive BHs

Ferreira+PRD96 (2017)083017; Boskovic+PRD98 (2018) 024037; Davoudiasl+PRL123 (2019)021102; Bar+ JCAP1907 (2019)045; GRAVITY MNRAS 489 (2019) 4606

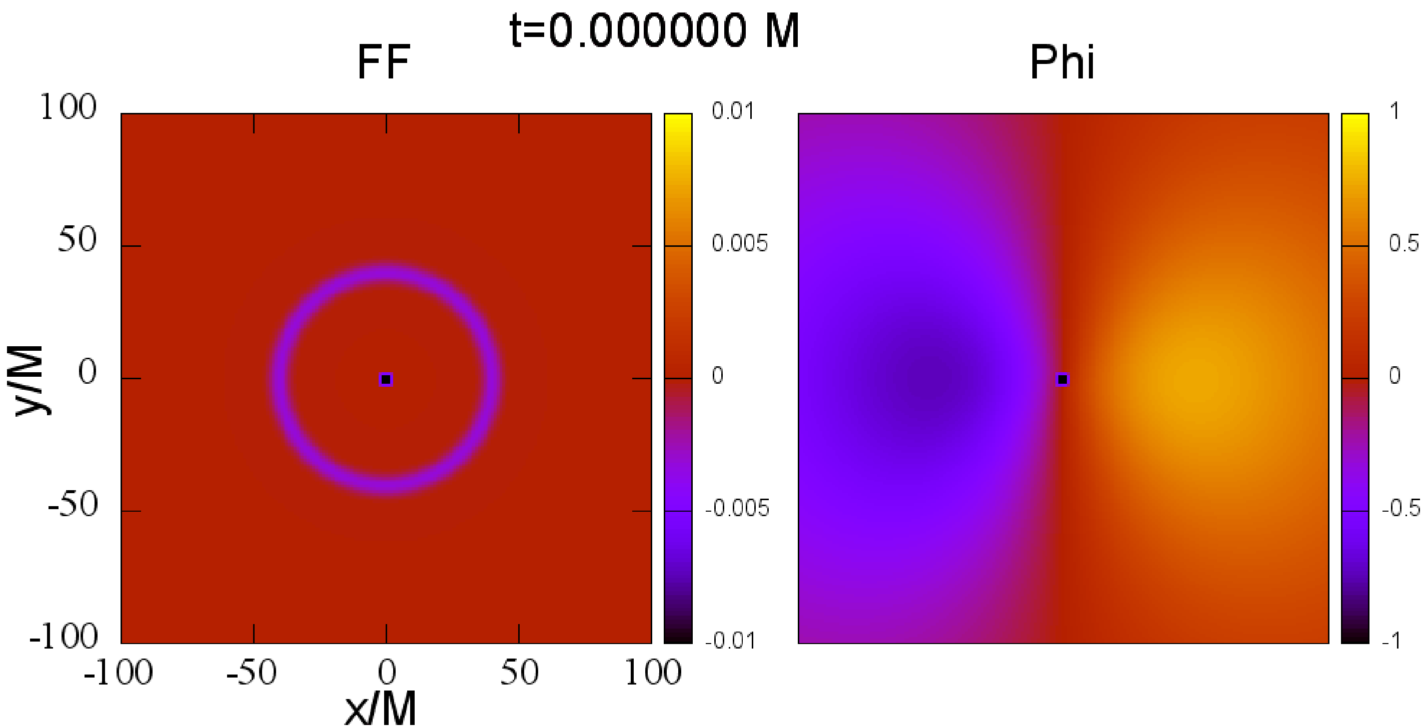
$$\mathcal{L} = \frac{R}{k} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}g^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi - \frac{\mu_S^2}{2}\Psi\Psi - \frac{k_{\text{axion}}}{2}\Psi\ {}^*F^{\mu\nu}F_{\mu\nu}$$

t=0.000000 M



Boskovic+ PRD99:035006 (2019); Ikeda+ PRL122:081101 (2019)

$$\mathcal{L} = \frac{R}{k} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}g^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi - \frac{\mu_S^2}{2}\Psi\Psi - \frac{k_{\mathrm{axion}}}{2}\Psi\,{}^*F^{\mu\nu}F_{\mu\nu}$$



Boskovic+ PRD99:035006 (2019); Ikeda+ PRL122:081101 (2019)

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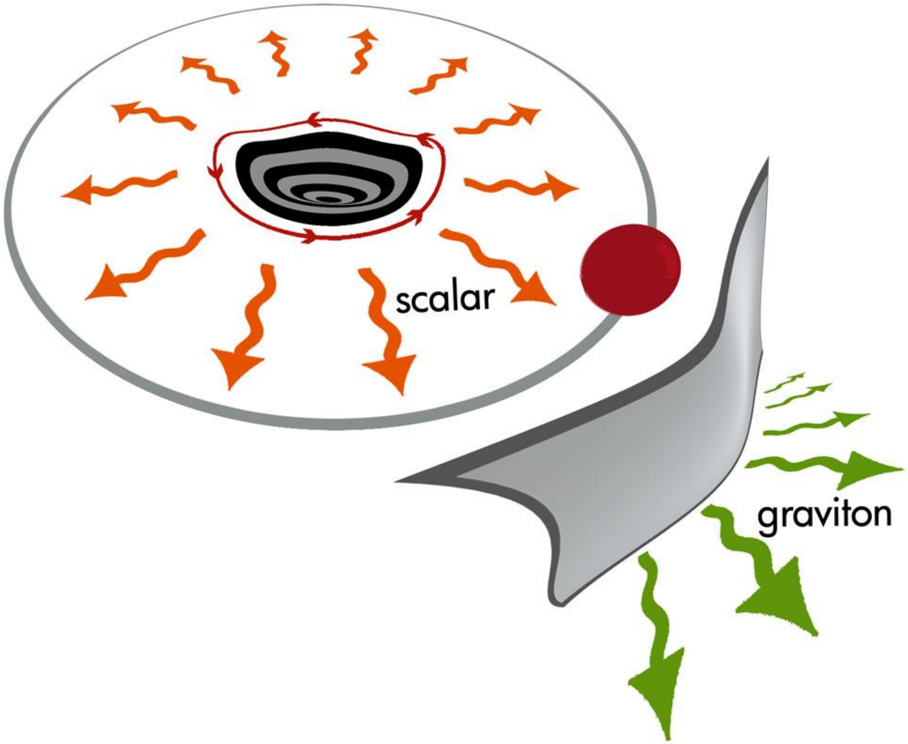
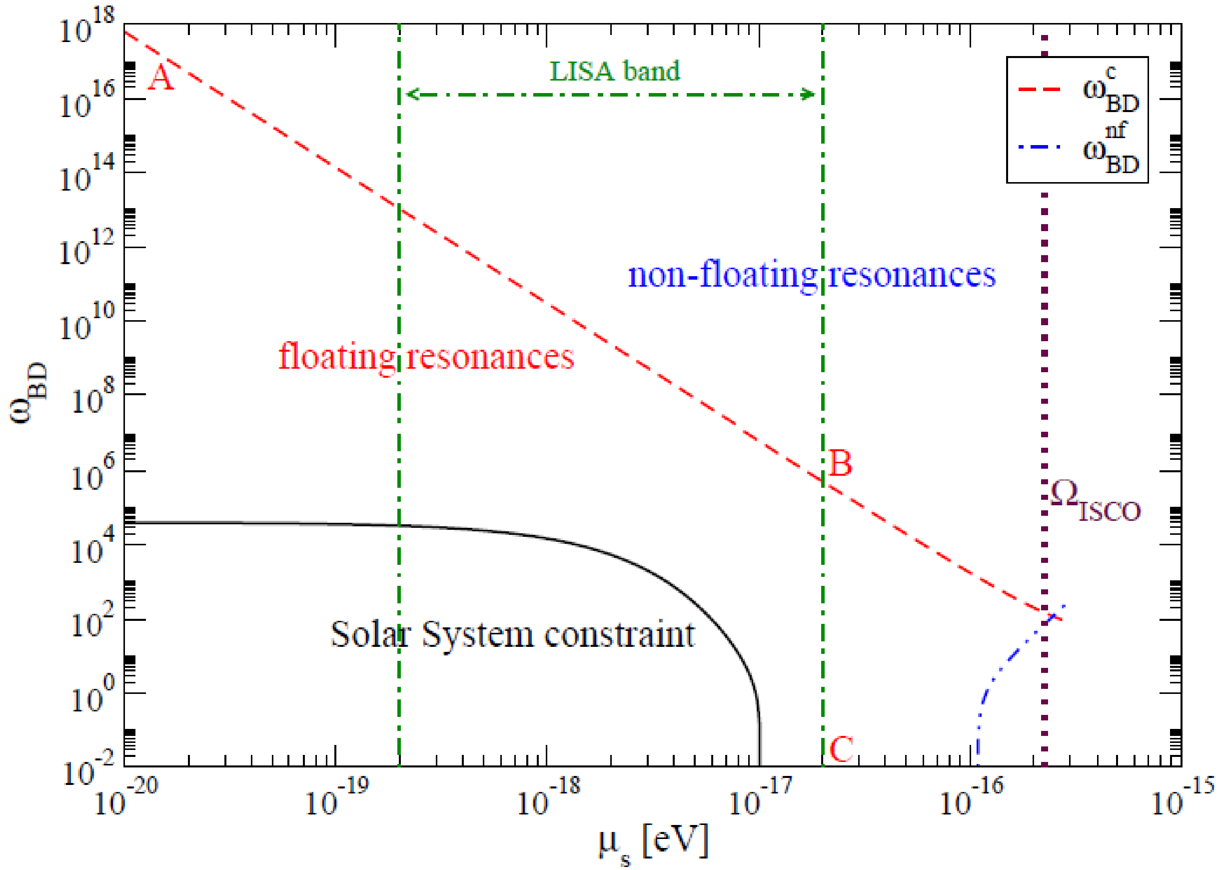
Ferreira+PRD96 (2017)083017; Boskovic+PRD98 (2018) 024037; Davoudiasl+PRL123 (2019)021102; Bar+ JCAP1907 (2019)045; GRAVITY MNRAS 489 (2019) 4606

Orbital motion of stars and planets I. Floating orbits

$$\left[\Box - \mu_s^2\right] \varphi = \alpha \mathcal{T}$$

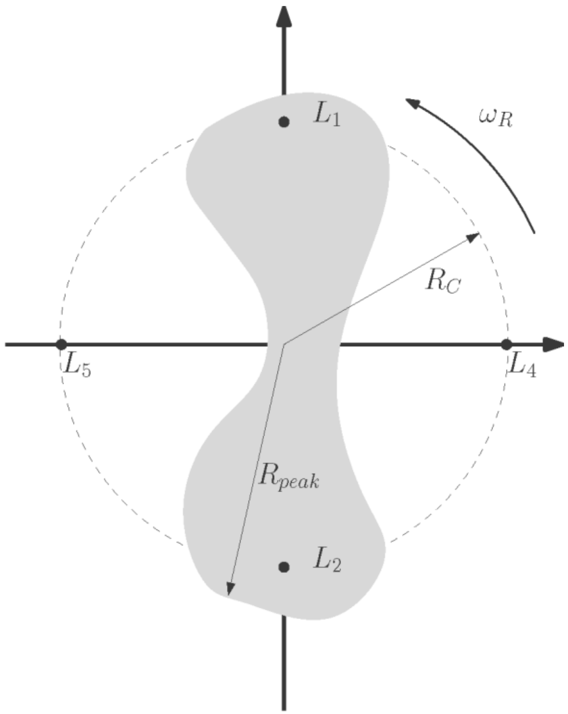


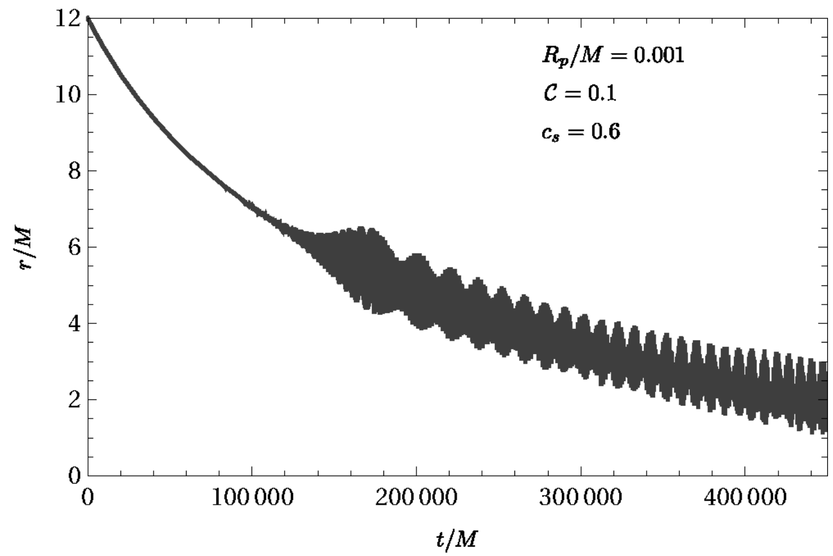
Cardoso + PRL107:241101 (2011); Yunes + PRD81, 084052 (2012);
Fujita + PRD95:044016 (2017); Zhang & Yang, arXiv:1808.02905



Yunes+ PRD81, 084052 (2012)

Orbital motion of stars and planets II. Minimal couplings





Macedo+ PRD96:083017 (2017); Boskovic+ PRD98:024037 (2018)

Also Khmelnitsky & Rubakov JCAP1402:019 (2014); Blas+ PRL 118 (2017) 261102

III. Transitions, disruption and floating by tides

$$V = -\theta\left(t - t_0\right) \frac{M_c\,\mu}{R} \sum_{|m|\leq 2} \frac{4\pi}{5} \left(\frac{r}{R}\right)^2 Y_{lm}^*\left(\theta_c, \phi_c\right) Y_{lm}\left(\theta, \phi\right)$$

$$\mathcal{H}\psi_n = E_n\psi_n$$
$$\mathcal{H} = \mathcal{H}_0 + V$$

Use standard perturbation theory:

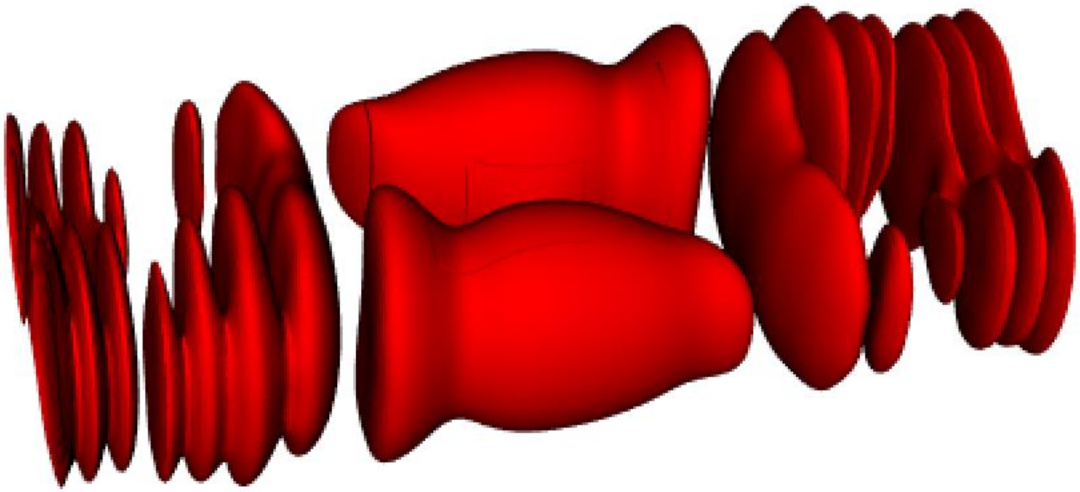
Find level transitions which lead to puffing up of cloud

Levels which change mulitpolar structure

Resonances and cloud destruction

Zhang+ PRD99 (2019) 064018; Baumann + PRD99 (2019) 044001; Cardoso+ (in progress)

Tidal disruption of clouds



Cardoso+ (in progress)

Energy extraction from black hole binaries?

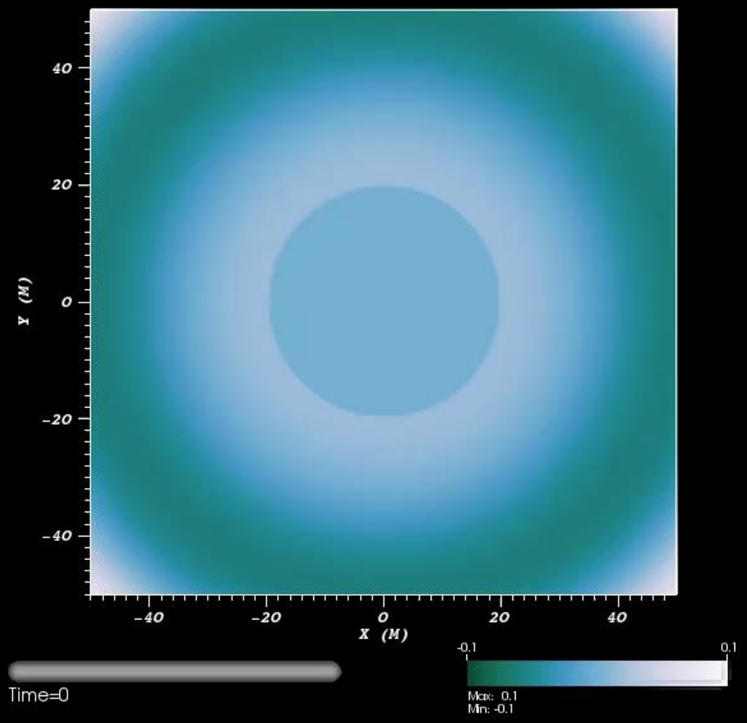
1. Superradiance, if individual black holes spin

2. Ergoregions in binaries?

3. Slingshot effect for massless waves

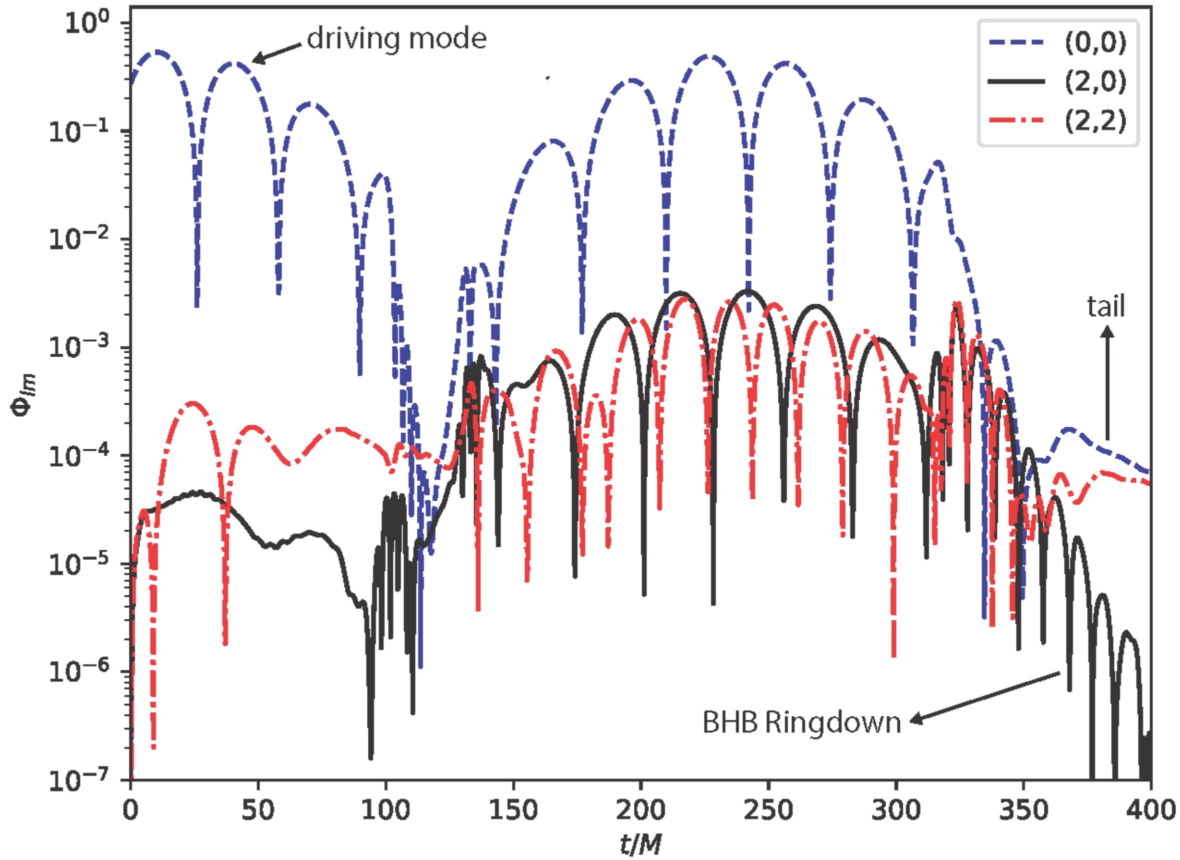
4. Parametric resonance? Fermi-like acceleration?

Gravitational molecules: a BH binary



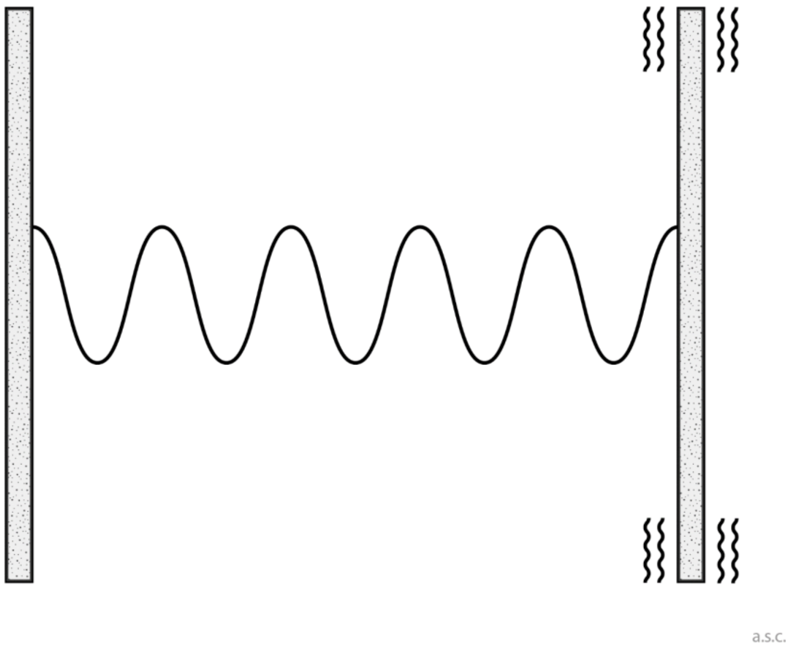
Bernard + PRD100 (2019) 044002 (and work in progress)

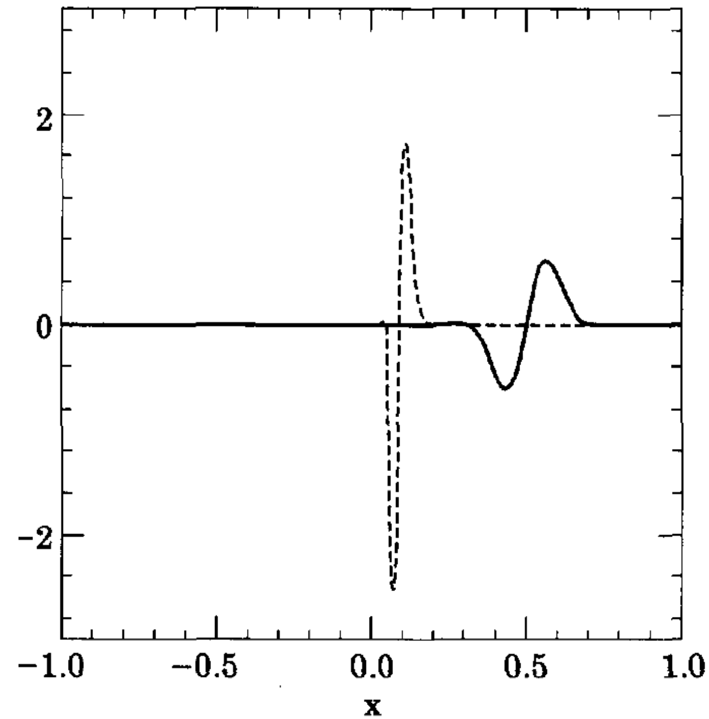
Spectroscopy of gravitational molecules



Global BHB modes may be resonantly excited

Bernard + PRD100 (2019) 044002 (and work in progress)





Cooper IEEE Trans. Ant. Propag. 1993

Conclusions: exciting times!

Gravitational wave astronomy *will* become a precision discipline, mapping compact objects throughout the entire visible universe.

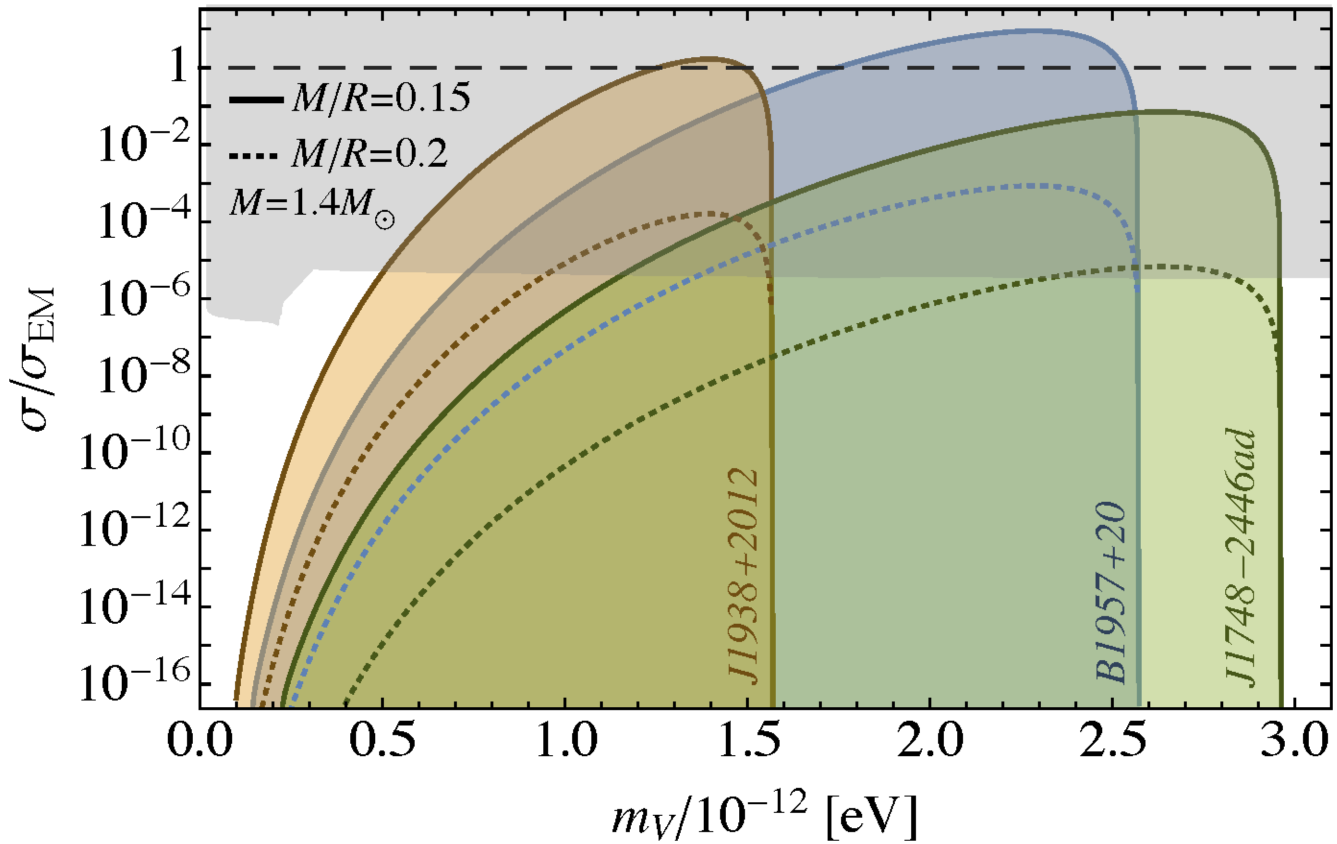
Black holes remain the most outstanding object in the universe. They respond in simple way to external perturbations, and may serve as seeds for atoms of fundamental light fields. Black hole and BH binary spectroscopy is an exciting tool to understand the content of our universe

“***After the advent of gravitational wave astronomy, the observation of these resonant frequencies might finally provide direct evidence of BHs with the same certainty as, say, the 21 cm line identifies interstellar hydrogen***”

(S. Detweiler ApJ 239:292 1980)

Thank you

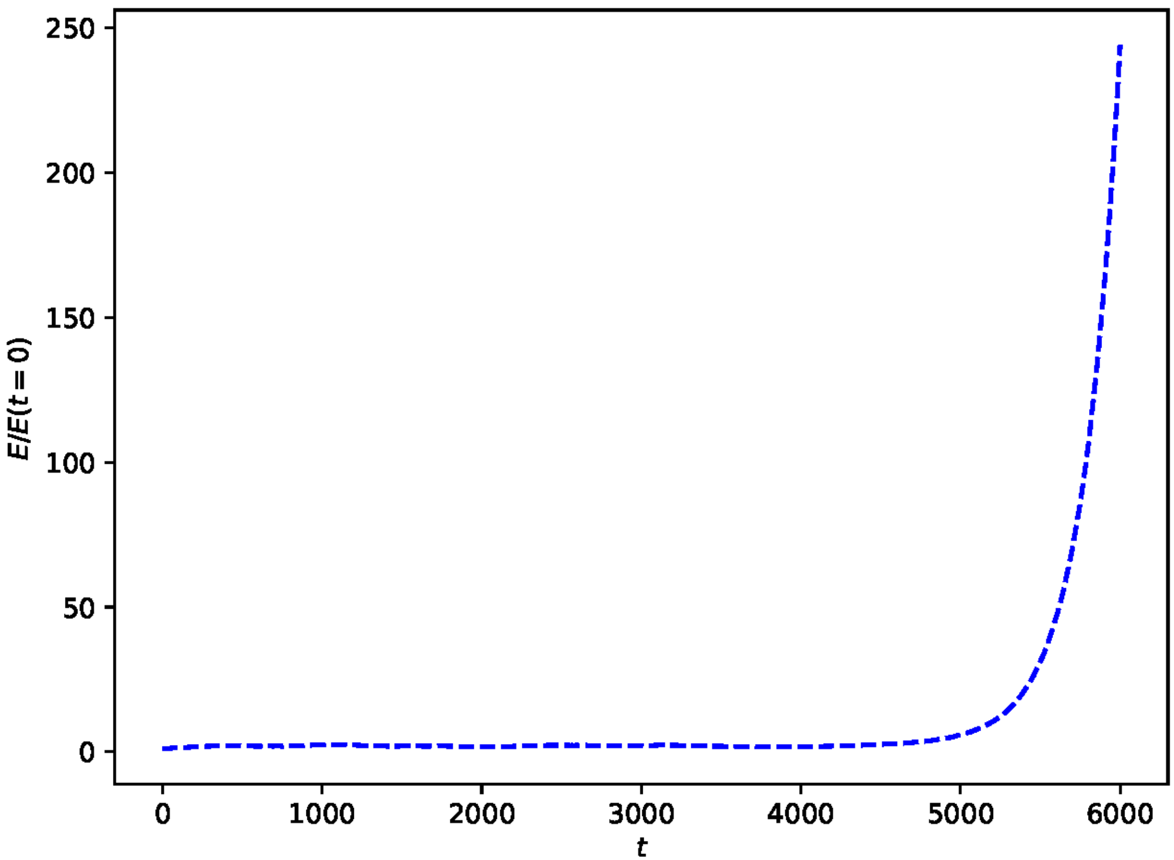
Stars?



Exclusion plots for known pulsars, based on measured spin-down rates. M is assumed to be M=1.4 $M_{\rm sun}$. Grey is excluded region from CMB distortion (photon->X depletion)

Cardoso+ PRD95: 124056 (2017); also Kaplan+arXiv:1908.10440

Energy extraction from binaries?

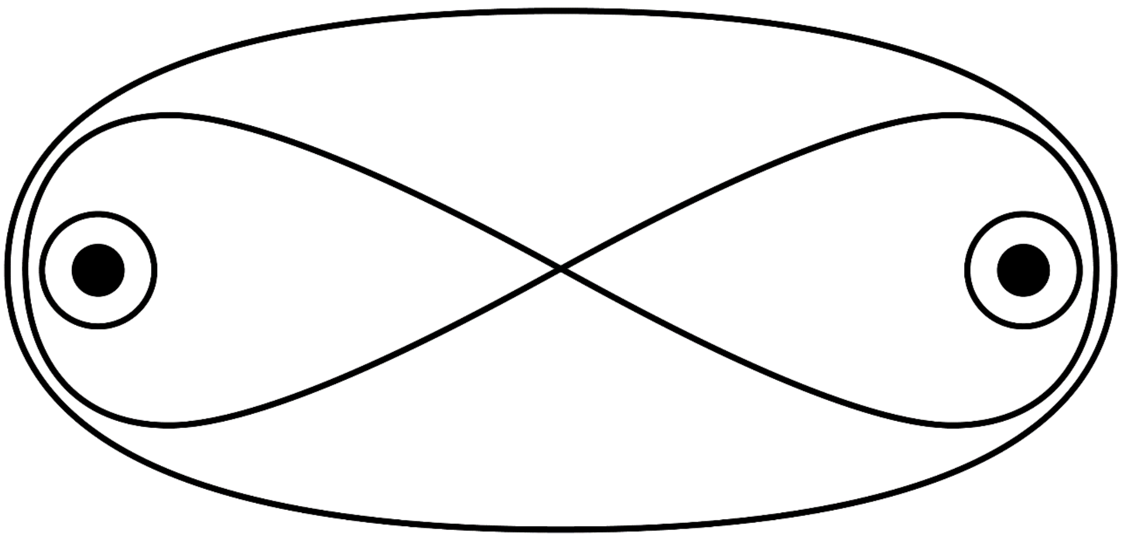


Bernard + arXiv:1905.05204 (and work in progress)

Gravitational molecules: a toy model

$$ds^2 = -\frac{dt^2}{U^2} + U^2\left(d\rho^2 + \rho^2 d\phi^2 + dz^2\right)$$

$$U(\rho,\ z) = 1 + \frac{M}{\sqrt{\rho^2 + (z-a)^2}} + \frac{M}{\sqrt{\rho^2 + (z+a)^2}}$$



Chandrasekhar PRSLA421:227 (1989); Assumpção+ PRD98: 064036(2018)

Gravitational molecules: a toy model

Change to prolate confocal elliptic coordinates

$$\rho^2 + (a-z)^2 = a^2(\chi+\eta)^2$$
$$\rho^2 + (a+z)^2 = a^2(\chi-\eta)^2$$

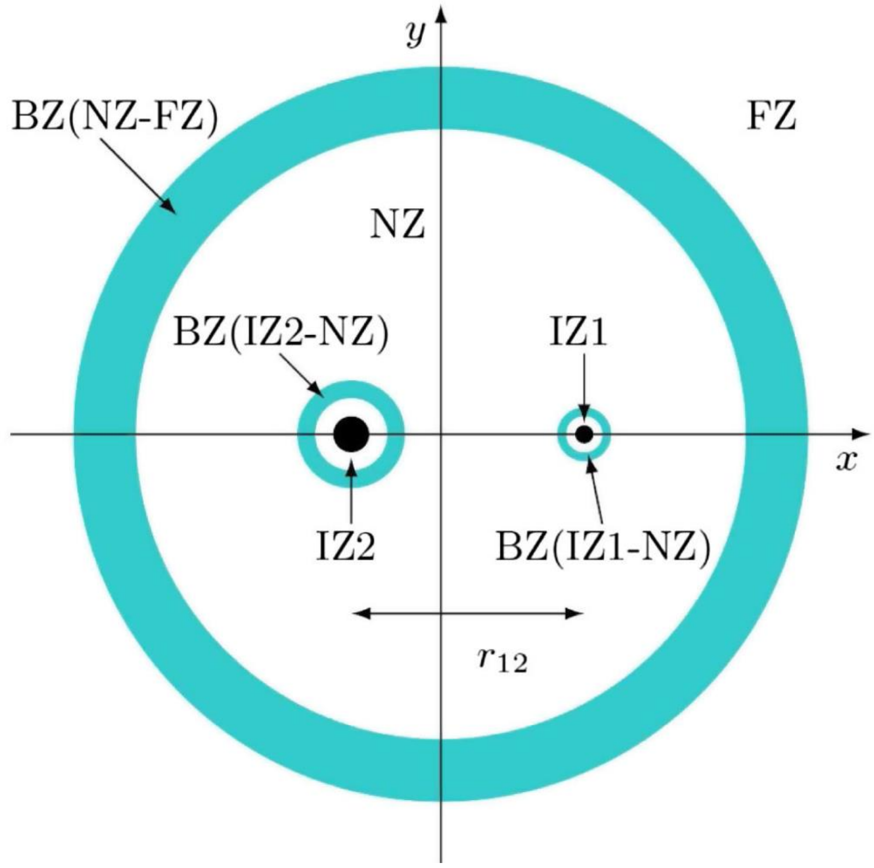
$$\partial_\eta\left((1-\eta^2)\partial_\eta S\right) + \left(-a^2\omega^2\eta^2 - \frac{m^2}{1-\eta^2} + \Lambda\right) S = 0$$
$$\partial_\chi\left((\chi^2-1)\partial_\chi R\right) + \left(a^2\omega^2\chi^2 + 8Ma\chi\omega^2 - \frac{m^2}{\chi^2-1} - \Lambda\right) R = 0$$

Klein-Gordon equation is identical to Schrodinger for Di-Hydrogen ionized molecule!

Bernard+ (2019)

for Hydrogen molecule see Burrau M7: 1 (1928); Wilson PRSLA118:635 (1929); Hylleraas ZfP71: 739 (1931)

Gravitational molecules: a real BH binary



Mundim+ PRD89: 084008 (2014); Bernard + (2019)

GWs and dark matter I

Dark matter is not a strong-field phenomenon, but GW observations may reveal a more "mundane" explanation in terms of heavy BHs

Bird + PRL116:201301 (2016)

II

Inspiral occurs in dark-matter rich environment and may modify the way inspiral proceeds, given dense-enough media: accretion and gravitational drag play important role.

Eda + PRL110:221101 (2013); Macedo + ApJ774:48 (2013)

DM II

Inspiral occurs in dark-matter rich environment and may modify the way inspiral proceeds, given dense-enough media: accretion and gravitational drag play important role.

Eda + PRL110:221101 (2013); Macedo + ApJ774:48 (2013); Barausse+PRD 2014

Self-gravity:

$$\rho_0 = 10^3 M_\odot \mathrm{pc}^{-3} \sim 10^4 \mathrm{GeV cm}^{-3}$$

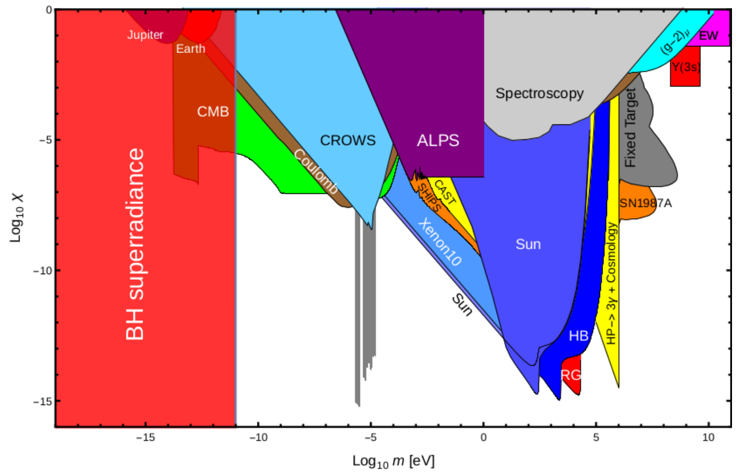
$$\frac{M^{\mathrm{DM}}_{\mathrm{inside\,r}}}{M_{\mathrm{BH}}} = 10^{-19}\left(\frac{M_{\mathrm{BH}}}{10^6 M_\odot}\right)^2\left(\frac{r}{100M}\right)^3 \frac{\rho_{\mathrm{DM}}}{\rho_0}$$

Accretion:

$$\dot{M}_{\mathrm{BH}} = \frac{16\pi G^2 M^2_{\mathrm{BH}}\rho_{\mathrm{DM}}}{v_{\mathrm{DM}} c^2} \quad \left(\dot{M} = \sigma\rho v\right)$$

$$\frac{\Delta M_{\mathrm{BH}}}{M_{\mathrm{BH}}} = 10^{-16}\left(\frac{M_{\mathrm{BH}}}{10^6 M_\odot}\right)\frac{\rho_{\mathrm{DM}}}{\rho_0}\frac{T}{1\,\mathrm{year}}\left(\frac{\sigma_v}{220\,\mathrm{Km/s}}\right)^{-1}$$

DM III. Light fields



Cardoso+ 2018, adapted from Sigl (2017) and Jaeckel arXiv:1303.1821

Interesting as effective description; proxy for more complex interactions; arise as interesting extensions of GR* *(BD or generic ST theories, f(R), etc)*

Bosons do exist (Higgs) and lighter versions may as well
Peccei-Quinn (interesting because not invented to solve DM problem), axiverse (moduli and coupling constants in string theory)

$$\mathcal{L} = \frac{R}{k} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}g^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi - \frac{\mu_{\rm S}^2}{2}\Psi\Psi - \frac{k_{\rm axion}}{2}\Psi\ {}^*F^{\mu\nu}F_{\mu\nu}$$

...and one or more could be a component of DM. *D. Marsh, Phys. Repts. 2016*

$$
\begin{aligned}
H_+^{(0)} &= -(1+c^2)\cos 2\psi \ , \\
H_+^{(1/2)} &= -\frac{s}{8}\frac{\delta m}{m}\left[(5+c^2)\cos\psi - 9(1+c^2)\cos 3\psi\right] \ , \\
H_+^{(1)} &= \frac{1}{6}\left[(19+9c^2-2c^4) - \eta(19-11c^2-6c^4)\right]\cos 2\psi \\
&\quad - \frac{4}{3}s^2(1+c^2)(1-3\eta)\cos 4\psi \ , \\
H_+^{(3/2)} &= \frac{s}{192}\frac{\delta m}{m}\left\{\left[(57+60c^2-c^4) - 2\eta(49-12c^2-c^4)\right]\cos\psi\right. \\
&\quad - \frac{27}{2}\left[(73+40c^2-9c^4) - 2\eta(25-8c^2-9c^4)\right]\cos 3\psi \\
&\quad \left. + \frac{625}{2}(1-2\eta)s^2(1+c^2)\cos 5\psi\right\} - 2\pi(1+c^2)\cos 2\psi \ , \\
H_+^{(2)} &= \frac{1}{120}\left[(22+396c^2+145c^4-5c^6) + \frac{5}{3}\eta(706-216c^2-251c^4+\right. \\
&\qquad \left. - 5\eta^2(98-108c^2+7c^4+5c^6)\right]\cos 2\psi \\
&\quad + \frac{2}{15}s^2\left[(59+35c^2-8c^4) - \frac{5}{3}\eta(131+59c^2-24c^4)\right. \\
&\qquad \left. + 5\eta^2(21-3c^2-8c^4)\right]\cos 4\psi \\
&\quad - \frac{81}{40}(1-5\eta+5\eta^2)s^4(1+c^2)\cos 6\psi \\
&\quad + \frac{s}{40}\frac{\delta m}{m}\left\{\left[11+7c^2+10(5+c^2)\ln 2\right]\sin\psi - 5\pi(5+c^2)\cos\psi\right.
\end{aligned}
$$